BU CAS CS 591 (Spring, 2003)
Computation and Deduction

Solution 2

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Solution of Exercise 2.9 Let us define FV and BV as follows.

$$\begin{array}{rcl} FV(x) & = & \{x\} \\ FV(\mathbf{lam}\, x.e) & = & FV(e)\backslash\{x\} \\ FV(e_1(e_2)) & = & FV(e_1) \cup FV(e_2) \end{array}$$

$$\begin{array}{rcl} BV(x) & = & \{x\} \\ BV(\mathbf{lam}\, x.e) & = & \{x\} \cup BV(e) \\ BV(e_1(e_2)) & = & BV(e_1) \cup BV(e_2) \end{array}$$

Let $V(e) = FV(e) \cup BV(e)$. Given two variables x and y, we define $(y/x)e$ as follows.

$$\begin{array}{rcl} (y/x)z & = & z \text{ if } x \neq z \\ (y/x)z & = & y \text{ if } x = z \\ (y/x)(\mathbf{lam}\, z.e) & = & \mathbf{lam}\, z.e \text{ if } x = z \\ (y/x)(\mathbf{lam}\, z.e) & = & \mathbf{lam}\, z.(y/x)e \text{ if } x \neq z \\ (y/x)(e_1(e_2)) & = & ((y/x)e_1)((y/x)e_2) \end{array}$$

We now define the α-equivalence relation as follows.

$$\frac{}{x =_\alpha x}$$

$$\frac{z \notin V(e) \cup V(e') \quad (z/x)e =_\alpha (z/x')e'}{\mathbf{lam}\, x.e =_\alpha \mathbf{lam}\, x'.e'}$$

$$\frac{e_1 =_\alpha e_1' \quad e_2 =_\alpha e_2'}{e_1(e_2) =_\alpha e_1'(e_2')}$$

Finally, we define the subsitution function as follows.

$$\begin{array}{rcl} [e/x]z & = & z \text{ if } x \neq z \\ [e/x]z & = & e \text{ if } x = z \\ [e/x](\mathbf{lam}\, y.e_0) & = & \mathbf{lam}\, z.[e/x](z/y)e_0, \\ & & \text{where } z \text{ is the first variable such that } z \notin FV(e) \cup V(e_0) \\ [e/x](e_1(e_2)) & = & ([e/x]e_1)([e/x]e_2) \end{array}$$

With this definition, we can now establish various properties for substitution.