

Development Separation in Lambda-Calculus

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Motivation for the Research

- To facilitate inductive reasoning on λ -terms
- To facilitate the encoding of λ -calculus in proof systems
 - Proofs based on structural induction on λ -terms are often amenable to such encoding

Some Notations (1)

λ -terms $M, N ::= x \mid \lambda x.M \mid M(N)$

- $\mathbf{FV}(M)$ denotes the set of free variables in M .
- A β -redex R is a λ -term of the form $(\lambda x.M_1)(M_2)$ and its contractum is $M_1[x := M_2]$.
- $M_1 \rightarrow_{\beta} M_2$ stands for 1-step β -reduction: M_2 is obtained from replacing a β -redex in M_1 with its contractum.
- $M_1 \rightarrow_{\beta}^n M_2$ stands for n -step β -reduction ($n \geq 0$).
- $M_1 \rightarrow_{\beta}^* M_2$ stands for multi-step β -reduction.
- $M_1 \xrightarrow{R}_{\beta} M_2$ stands for 1-step β -reduction where a particular occurrence of the redex R in M_1 is reduced.

Some Notations (2)

Given a finite β -reduction sequence $\sigma : M \rightarrow_{\beta}^* N$, we use $\sigma(M)$ for N and $|\sigma|$ for the number of β -reduction steps in σ .

- Given $\sigma_1 : M_1 \rightarrow_{\beta}^* M_2$ and $\sigma_2 : M_2 \rightarrow_{\beta}^* M_3$, we use $\sigma_1 + \sigma_2$ for the concatenation of σ_1 and σ_2 .
- $[R_1] + [R_2] + \dots + [R_n]$ stands for a β -reduction sequence of the following form:

$$M_1 \xrightarrow{R_1}_{\beta} M_2 \xrightarrow{R_2}_{\beta} \dots \xrightarrow{R_n}_{\beta} M_{n+1}$$

So each finite σ is always of the form $[R_1] + [R_2] + \dots + [R_n]$.

Residuals of β -redexes (1)

Let \mathcal{R} be a set of β -redexes in M_1 . Assume $M_1 \xrightarrow{R}_\beta M_2$ for $R = (\lambda x.M)(N)$. Then this β -reduction step affects β -redexes R' in \mathcal{R} in the following ways:

- R' is R . Then R' has no residual in M_2 .
- R' is in N . Then all copies of R' in $M[x := N]$ are residuals of R' .
- R' is in M . Then the λ -term $R'[x := N]$ in M_2 is the residual of R' .
- R' contains R . Then the residual of R' in M_2 is the term obtained from replacing R with its contractum in R' .
- R' and R are disjoint. Then R' is not affected and is its own residual.

Residuals of β -redexes (2)

- We use $\mathcal{R}/[R]$ for the set of residuals of all β -redexes in \mathcal{R} after R is reduced.
- The residual relation is transitive. Given a β -reduction sequence $\sigma = [R_1] + [R_2] + \dots + [R_n]$, we use \mathcal{R}/σ for the residuals of all the β -redexes in \mathcal{R} under σ , which is formally defined as:

$$\mathcal{R}/\sigma = (\dots ((\mathcal{R}/[R_1])/[R_2]) \dots)/[R_n]$$

Involvement

Given $M \rightarrow_{\beta}^* M_1$ and $\sigma : M_1 \rightarrow_{\beta} M_2 \rightarrow_{\beta} \dots$, we say that a redex R in M is involved in σ if a residual of R in M_1 is reduced in σ .

Developments

Let \mathcal{R} be a set β -redexes in M . A β -reduction sequence σ from M is a development of \mathcal{R} if each β -redex reduced in σ is either in \mathcal{R} or a residual of some β -redex in \mathcal{R} .
Furthermore, σ is a complete development of \mathcal{R} if \mathcal{R}/σ is empty.

Development Separation

Given a λ -term M , $M[x \dots, x]_x$ is a representation of M in which all the free occurrences of x in M are enumerated from left to right in $[x, \dots, x]$. We write $M[N_1, \dots, N_n]_x$ for the λ -term obtained by substituting N_i for the i th free occurrence of x in M for $i = 1, \dots, n$.

Let $M = (\lambda x.M_1)(M_2)$, \mathcal{R}_1 be a set of β -redexes in M_1 , \mathcal{R}_2 be a set of β -redexes in M_2 and $\mathcal{R} = \{M\} \cup \mathcal{R}_1 \cup \mathcal{R}_2$. For every development σ of \mathcal{R} in which M is involved, $\sigma(M)$ is of the following form:

$$\sigma_1(M_1)[\sigma_{2,1}(M_2), \dots, \sigma_{2,n}(M_2)]_x,$$

where σ_1 is a development of \mathcal{R}_1 from M_1 and $\sigma_{2,i}$ are developments of \mathcal{R}_2 from M_2 for $i = 1 \leq i \leq n$.

Canonical and Standard Developments

Let $\sigma = [R_1] + \dots + [R_n]$ be a development of \mathcal{R} . If R_j are not residuals of any β -redexes containing R_i for all $1 \leq i < j \leq n$, then σ is a canonical development.

Furthermore, if R_j are not residuals of any β -redexes to the left of R_i for all $1 \leq i < j \leq n$, then σ is a standard development.

Canonicalization of Developments

For every development $\sigma : M \rightarrow_{\beta}^* N$ of \mathcal{R} , there exists a canonical development $\text{cad}(\sigma) : M \rightarrow_{\beta}^* N$ such that for every $R \in \mathcal{R}$, R is involved in σ if it is involved in $\text{cad}(\sigma)$.

Proof of Canonicalization of Developments

The proof is by structural induction on M . We present an interesting case where $M = (\lambda x.M_1)(M_2)$ and M is involved in σ . By Development Separation, $\text{sep}(\sigma)$ is of the following form:

$$[M] + \sigma_1[x := N] + \sigma_{2,1} + \dots + \sigma_{2,n}$$

where σ_1 is a development from M_1 and $\sigma_{2,i}$ are developments from M_2 for $1 \leq i \leq n$. By induction hypothesis, we can define $\text{cad}(\sigma)$ as follows:

$$\text{cad}(\sigma) = [M] + \text{cad}(\sigma_1)[x := N] + \text{cad}(\sigma_{2,1}) + \dots + \text{cad}(\sigma_{2,n})$$

Clearly, $\text{cad}(\sigma)$ is canonical since both $\text{cad}(\sigma_1)$ and $\text{cad}(\sigma_{2,i})$ are canonical for $1 \leq i \leq n$. Assume that $R \in \mathcal{R}$ is involved in $\text{cad}(\sigma)$. Then it can be readily verified that R is involved in $\text{sep}(\sigma)$. Hence, R is also involved in σ .

Standardization of Developments

For every development $\sigma : M \rightarrow_{\beta}^* N$ of \mathcal{R} , there exists a standard development $\text{std}(\sigma) : M \rightarrow_{\beta}^* N$ such that for every $R \in \mathcal{R}$, R is involved in σ if it is involved in $\text{std}(\sigma)$.

Proof: This follows from canonicalization of developments immediately: $\text{std}(\sigma)$ is obtained by a proper reshuffling of $\text{cad}(\sigma)$.

Standardization Lemma

Given $\sigma = \sigma_1 + \sigma_2$, where σ_1 is a standard development of \mathcal{R} and σ_2 is a standard β -reduction sequence, we can construct a β -reduction sequence $\text{std}_2(\sigma_1, \sigma_2)$ which standardizes σ .

Proof of Standardization Lemma (1)

By Development Separation, we have that the function std is defined on all developments. Let us define $\text{std}_2(\sigma_1, \sigma_2)$ and prove that $\text{std}_2(\sigma_1, \sigma_2)$ standardizes $\sigma_1 + \sigma_2$ by induction on $\langle |\sigma_2|, |\sigma_1| \rangle$, lexicographically ordered. Clearly, for σ_1, σ_2 , $\text{std}(\sigma_1, \emptyset)$ and $\text{std}(\emptyset, \sigma_2)$ can be defined as σ_1 and σ_2 , respectively. We now assume $\sigma_1 = [R_1] + \sigma'_1$ and $\sigma_2 = [R_2] + \sigma'_2$, and we have two cases.

Proof of Standardization Lemma (2)

- R_2 is a residual of some β -redex in \mathcal{R} that is to the left of R_1 . Hence, $\sigma_1 + [R_2]$ is a development. We define $\text{std}_2(\sigma_1, \sigma_2)$ as follows:

$$\text{std}_2(\sigma_1, \sigma_2) = \text{std}_2(\text{std}(\sigma_1 + [R_2]), \sigma'_2)$$

Assume that $R \in \mathcal{R}$ is involved in $\text{std}_2(\sigma_1, \sigma_2)$. Then by induction hypothesis, R is involved in $\text{std}(\sigma_1 + [R_2]) + \sigma'_2$. This implies that R is involved in $\sigma_1 + [R_2] + \sigma'_2 = \sigma_1 + \sigma_2 = \sigma$.

- R_2 is not a residual of any β -redex in \mathcal{R} that is to the left of R_1 . Then we define $\text{std}_2(\sigma_1, \sigma_2)$ as follows:

$$\text{std}_2(\sigma_1, \sigma_2) = [R_1] + \text{std}_2(\sigma'_1, \sigma_2)$$

Standard β -Reduction Sequences

Given a β -reduction sequence σ of the following form:

$$M_1 \xrightarrow{R_1}_{\beta} M_2 \xrightarrow{R_2}_{\beta} \cdots \xrightarrow{R_n}_{\beta} M_{n+1}$$

we say that σ is standard if for all $1 \leq i < j \leq n$, R_j is not a residual of some β -redex to the left of R_i . We say that $\sigma_s : M \xrightarrow{*}_{\beta} N$ standardizes σ if σ_s is a standard β -reduction sequence and for every R in M that is involved in σ_s , R is also involved in σ .

Standardization Theorem

For every β -reduction sequence σ , we can construct a β -reduction sequence $\text{std}_1(\sigma)$ that standardizes σ .

Proof: Let us define std_1 as follows:

$$\text{std}_1(\emptyset) = \emptyset \quad \text{std}_1([R] + \sigma) = \text{std}_2([R], \text{std}_1(\sigma))$$

By Standardization Lemma, $\text{std}_1(\sigma)$ standardizes σ .

$$\mu(M)$$

If M is strongly normalizing, let $\mu(M)$ be the least natural number such that $|\sigma| \leq \mu(M)$ holds for each β -reduction sequence from M . Otherwise, let $\mu(M) = \infty$.

Proposition on μ

Assume $M \xrightarrow{R}_\beta M'$, where $R = (\lambda x.N_1)(N_2)$ is the leftmost β -redex in M . Then $\mu(M) \leq \mu(M') + \mu(N_2)$ holds.

The proof is straightforward.

Normalization Lemma

Given a λ -term M , let $\Lambda(M)$ be the longest leftmost β -reduction sequence from M , which may be of infinite length. Assume that $\sigma : M \rightarrow_{\beta}^* M'$ is a standard development. If $|\Lambda(M')| < \infty$, then $|\Lambda(M')| \leq |\Lambda(M)| < \infty$ holds.

Proof of Normalization Lemma (1)

The proof proceeds by induction on $\langle |\Lambda(M')|, |\sigma| \rangle$, lexicographically ordered. If M is in β -normal form, then $M' = M$ and we are done. We now assume $M \xrightarrow{R_l}_{\beta} M_l$, where R_l is the leftmost β -redex in M . Then $\Lambda(M) = [R_l] + \Lambda(M_l)$. We have two cases as follows.

Proof of Normalization Lemma (2)

- R_l is involved in σ . Since σ is standard, σ is of the form $M \xrightarrow{R_l}_\beta M_l \xrightarrow{\sigma'}^*_\beta M'$ for some standard development σ' . Since $|\sigma'| < |\sigma|$ holds, we have $|\Lambda(M')| \leq |\Lambda(M_l)| < \infty$ by induction hypothesis. Hence $|\Lambda(M')| \leq |\Lambda(M)| < \infty$ holds.
- R_l is not involved in σ . Then R_l has a residual R'_l in M' , which also happens to be the leftmost β -redex in M' . Then $\sigma + [R'_l]$ is a development of $\mathcal{R} \cup \{R_l\}$. Hence $\text{std}(\sigma + [R'_l]) = R_l + \sigma'$ for some $\sigma' : M_l \xrightarrow{\sigma'}^*_\beta M'_l$, which is a standard development of $\mathcal{R}/[R_l]$. Assume $M' \xrightarrow{R'_l}_\beta M'_l$. Then $|\Lambda(M'_l)| < |\Lambda(M')|$ holds. By induction hypothesis, we have $|\Lambda(M'_l)| \leq |\Lambda(M_l)| < \infty$. This yields that $|\Lambda(M')| = 1 + |\Lambda(M'_l)| \leq 1 + |\Lambda(M_l)| = |\Lambda(M)| < \infty$.

Normalization Theorem

If M can be reduced to a normal form, then $|\Lambda(M)| < \infty$ holds.

Proof: By Normalization Lemma, the theorem follows from straightforward induction on the length of σ .

Related Work

- Parallel β -reductions are complete developments. Therefore, it is not surprising that the work in [Takahashi] can also be done in our setting.
- On the other hand, Takahashi's method can clearly be used to establish various lemmas in this paper (after they are properly formulated in terms of parallel β -reductions). This can probably be described as *separating parallel β -reductions (from other β -reductions)*.

Conclusion

We have demonstrated some interesting uses of the *development separation* lemma, proving by structural induction on λ -terms that developments are Church-Rosser and can be standardized.

The Church-Rosser theorem in λ -calculus follows immediately. Also, we have employed the technique of development separation in establishing structurally inductive proofs for the standardization theorem, the conservation theorem and the normalization theorem.