Problem 1.
(a) Similarly to problems 9-10 in the past Problem Set 0, is IMPLIES (⇒) gate universal? Prove you answer (e.g., by reduction to a statement/problem in Problem Set 0).
(b) How about a set of gates consisting of XOR and NOT?
(c) How about a set of gates consisting of XOR and AND?

Problem 2. We have discussed, violation of all of the following 5 properties is necessary and sufficient for a set of gates to be universal:

1. Linearity: \( f(x_1, \ldots, x_k) \) is linear if \( \exists I \subseteq \{1, \ldots, k\} \) such that \( \forall i \in \{1, \ldots, k\} \) and \( \forall c_j \neq i \) \( f(x_1 = c_1, \ldots, x_i = 0, \ldots, x_k = c_k) \neq f(x_1 = c_1, \ldots, x_i = 1, \ldots, x_k = c_k) \iff i \in I. \)
2. Self-duality: see the slide on the web
3. Monotonicity: \( f(x_1, \ldots, x_k) \) is monotone if \( \forall i \) and \( \forall c_j \neq i \) \( f(x_1 = c_1, \ldots, x_i = 0, \ldots, x_k = c_k) \leq f(x_1 = c_1, \ldots, x_i = 1, \ldots, x_k = c_k) \)
4. 0-preservation: \( f(0, \ldots, 0) = 0 \)
5. 1-preservation: \( f(1, \ldots, 1) = 1 \)

(a) For each of these properties give a non-universal gate violating this property.

(b) Prove that each property is closed under composition: that is show that if you have a set of gates all of which obey the given property, then any circuit you build out of these gates will also obey this property.
Start with the easy ones: you can do this exercise for any four of the five properties; the fifth (your choice!) is extra credit.

(c) Prove that any set of gates all of which obey at least one (any!) of these properties cannot be universal. Can there be a universal set such that every gate in it obeys at least one of these properties?

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1We can divide the input variables of \( f \) into the “important” and “ignorable” — \( f() \) depends only on the “important” variables and ignores the “ignorable” ones. \( I \) is the set of indices of the “important” variable. Linearity means that for any “important” variable, changing its value while keeping the rest of the variables fixed (at any values) always changes the output; and changing an “ignorable” value never changes the output. \( A \iff B \) means \( (A \Rightarrow B) \land (B \Rightarrow A) \)

2The part (b) and (c) together thus prove the “necessary” part of the above condition for universality. Proving the “sufficient” clause is much harder.
Problem 3. Consider a set $S$ of $k$ elements. Define Power set $P(S)$ of $S$ to be the set of all its subsets: $P(S) = \{T : T \subseteq S\}$ (this is just as discussed in class).

Now, define $P_2(S)$ to be the set of all multi-sets$^3$ containing at most two (identical) instances of each element of set $S$.$^4$

Similarly we can define $P_b(S)$ as the set of all multi-sets containing at most $b$ instances of each element. (Thus $P(S) = P_1(S)$.)

(a) List all the multi-sets of $P_2(\{\text{apple, orange}\})$. How many of them are there (i.e., what is the size $|P_2(\{\text{apple, orange}\})|$ of $P_2(\{\text{apple, orange}\})$)?

Let $S$ contain $k$ elements (written as $|S| = k$).

(b) What is $|P_1(S)|$?

(c) What is $|P_2(S)|$?

(d) What is $|P_b(S)|$?

Hint: For each of the above think of how each multi-set can be represented (by a “characteristic vector”).

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$^3$A multi-set is different from a set in that each element can be present at most once in a set, but more than once in a multi-set. The multiple instances of an element in a multi-set are still indistinguishable. So, if I have a multi-set consisting of five instances of an apple, than there is only one instance to select one apple from this multi-set.

$^4$For example, if $S = \{\text{orange, apple, banana}\}$, then $P_2(S)$ will contain all elements of $P(S)$ (i.e., $P(S) \subset P_2(S)$). But in addition, $P_2(S)$ contains, for example, a multi-set containing one orange, no apples, and two bananas; and another multi-set (different from the previous) contains two oranges, no apples, and one banana; and yet another one: 2 oranges, 2 apples and 2 bananas (this is the largest multi-set in $P_2(S)$).