MA/CS 109 Lecture 10

End of Population Models
Start of Probability and Statistics
Recall our Logistic model

\[ R(t+1) = k \times R(t) \times (1 - \frac{R(t)}{C}) \]

Examples of this model are shown below for \( C=4 \)

And two different values of \( k \).

Left is \( k=1.6 \), right is \( k=3.3 \). Note the population grows more quickly when small, but still “crashes” at 4.
What is the predicted population? Say with $R(0)=0.3$ as before? (The vertical and horizontal scales are the same on both graphs.)

$k=1.6$, $C=4$, $R(0)=0.3$  
k=3.3, $C=4$, $R(0)=0.3$
Let’s try the same $k=3.3$ and $C=4$ but $R(0)=3.9$ (almost the crash value).

Again we see that the population crashes to near zero, then grows and falls into the oscillating periodic behavior.
So as the growth rate constant increases, we not only see faster initial growth, we see different possible long-term behaviors.

The logistic model has more possible predictions as \( k \) varies.

What happens if \( k \) gets even bigger?
Let’s try k=4 (same C=4)…The “one year in the future” graph doesn’t look that much different than k=3.3, C=4…
Try the same initial data $R(0)=0.3$ and see what is predicted in the future...Nope...

This is new! Maybe it just hasn’t “settled down” yet. Let’s look at this prediction farther into the future.
Same $k=4$, $C=4$, $R(0)=0.3$, but time up to 50

Has not settled down! Population repeatedly grows quickly, then crashes, but never exactly the same values... This is very strange behavior!
Even stranger behavior... Same $k=4$ and $C=4$ but two different initial populations $R(0)=0.3$ and $R(0)=0.31$

Orange is $R(0)=0.31$, blue is $R(0)=0.3$. The predicted populations are very close up to year 6, but then the separate dramatically.
We have seen this before!!

Remember that a small error leads to quickly to a big difference in prediction for the Exponential Growth Model. There it didn’t matter much because we were predicting a population explosion in any case…

But here, where the prediction is of rising and crashing populations, the difference in prediction is dramatic.
Even a tiny change in the value of $R(0)$ leads, relatively quickly to a big change in prediction.

Blue $R(0)=0.3$, Orange $R(0)=0.30001$. After 16 years a slight difference in prediction, after 20 years a huge difference in prediction!
What have we seen??
When \( k=4 \) (and \( C=4 \)) the logistic model makes very strange predictions...

1. The long-term behavior is wild, irregular, jumpy, “Chaotic”...

2. Even a tiny change in initial population \( R(0) \) leads to a radically different population prediction in relatively short time, or, “errors” or changes grow exponentially.

These are the hallmarks of a “Chaotic” system—the “New Science” of “Chaos Theory”
Pessimist says

For \( k=4 \) (and \( C=4 \)) this model is useless...the prediction oscillate wildly AND any tiny error in the initial value \( R(0) \) (inevitable in the real world) lead to a wildly different prediction very quickly.
Optimist says

This is AMAZING! The simple Logistic model with 
k=4 and C=4 predicts really wild long-term behavior of the population AND a tiny change in initial value 
R(0) leads quickly to a very different prediction!

The real world is like that—systems like the weather and the stock market have wild behavior and it seems in life a small change can lead to a huge difference very quickly AND the simple Logistic model predictions behave like that!!
So complicated behavior might be happening for simple reasons! If the simple Logisitc model can predict complicated behavior, maybe the complications we see in the world around us, at least sometimes, have simple explanations!!

There is hope that we can understand how things like the weather and the human brain (even if we can not predict them far in the future with any accuracy....)
This is the science of “Chaos Theory”—It is a very optimistic view of the world. We should start studying complicated things by making simple models.

This is also unique in your study of Math—everything you have learned up until now was discovered or invented by a dead white European male over 300 years ago...But Chaos theory has its roots about 100 years ago and the chaos in the Logistic model was first studied in this way in 1976.
The most widely accepted definition of Chaotic system was developed by Prof. Robert Devaney who you might see around the Math Department talking to Prof. Nancy Kopell who has won Guggenheim and MacArthur “Genius” Fellowships for her work applying these ideas to models of nerve and brain cells….

You’ve made it to 21st century Mathematics.
If we can’t predict the future exactly...

So one of the lessons of Chaos Theory is that even if we have a simple model that accurately reflects how a system changes over short time, it may be impossible to predict very far in the future...

We will never be able to predict the weather (for example, if it will rain or not) a month in advance.
Why?

Because a butterfly flapping its wings an extra time in California could makes a tiny change in initial conditions—which will grow exponentially over time and in a month could impact the weather in Boston.
That doesn’t mean we give up!!!

It does mean we change the question. Instead of trying to predict the exact weather a month away, we should instead try to predict a probability of rain or the expected average rain under different conditions...we can’t say for sure if it will rain, but we do know that rain is much more likely than snow in Boston in September.

We might still predict the climate, even if we can’t predict the weather.
Transition...

So we need to have the tools to ask and answer questions in a quantitative way about things that we can never be sure of...

We need a way to “quantify our uncertainty”.

These are the goals of Probability and Statistics.
Probability and Statistics are members of the Mathematical Sciences. Certain aspects of these fields fall under “pure” mathematics and they originally developed out of mathematics by mathematicians.

In recent years these fields have grown both large and specialized. Many colleges and universities have separate departments of Statistics (BU even has a Department of Bio-Statistics at the Medical School).
However, Probability and Statistics share the methodology, employ many of the ideas and share the reliance on “proof” as the gold standard for what is known and understood.

We start with probability this week, then discuss inferential statistics in the following two weeks.
Suppose I flip a coin...

If the coin is a “fair” coin, then we say that the probability of heads is $1/2$.
And the probability of tails is $1/2$.
Everybody knows this...

But what does this mean?
This is not as easy a question as might first appear.

The flight of the coin is governed by the laws of physics, just like everything else.

If we know the initial velocity when the coin leaves my fingers and where it will be caught, then we should be able to predict exactly if it come up heads or tails...flipping a coin is a “deterministic” process.
On the other hand, I can not successfully predict how the coin will land—heads or tails. (Why? Because a tiny change in my throw can change the outcome—just like the Logistic model.)

So, even though the coin follows the laws of physics, there is uncertainty about what will happen.

Probability gives us a way to “quantify our uncertainty”. That is, it gives us a way to talk about things we are not sure about in a quantitative way.
Another way to think about this is that sometimes we have a little bit of information—that information may not be enough to let us precisely predict what will happen, that is, rule out all but one possible future, but it might still be useful.

Probability is a way to quantify this limited information.
Interpretation of probability

For our purposes, we say that a value of a probability of an event is the “long-range frequency of the event”—that is, the fraction of the time that the event will occur if we repeat the experiment many, many times.

So saying the probability of heads is ½ is saying that if I flip a coin many many many many times, then I expect that (at least pretty close to) ½ of the times it will come up heads.
This is not completely satisfying...suppose I don’t flip the coin many, many times. Suppose I only flip it once...does “probability = ½” still have meaning?

We need the “model building” step—we need to be precise about what we mean when we say “probability is ...”. Then we can see if our notions of probability relate to reality.

Vocabulary and Axioms for probability:
**Outcome** = possible result of an experiment or action. For example, when flipping a coin, the outcomes are Head (H) and Tails (T).

**Sample space** = the set of all possible outcomes of an experiment or action. For coin flip the sample space is \{H,T\} since these are the only things that can happen. If we flip a red and a blue coin the sample space is (red listed first) \{(H,H), (H,T), (T,H), (T,T)\}

**Event** = Subset of the possible outcomes. So for flipping a coin once, \{H\} is an event. For flipping a coin twice getting H first is an event, the event \{(H,H), (H,T)\}
Don’t be intimidated by the vocabulary (...and don’t use it to intimidate others).

Probability is a way to assign numbers to events. This assignment satisfies the following rules or axioms:

1. Every event is assigned a number between 0 and 1 (with both 0 and 1 possible).
2. The probability of the event that is the entire sample space is 1.
These axioms are completely reasonable. For axiom 1: If a probability is supposed to represent the fraction of the times a particular event occurs if an experiment is repeated many times, then the largest it can be is 1 (always happens) and the smallest it can be is 0 (never happens). Everything else is in between 0 and 1.

For axiom 2: The entire sample space is, by definition, the set of every possible outcome (everything that can happen). So what happens when we do the experiment is one of the outcomes, the probability of that thing being in the sample space is 1.
We only need one more axiom, but it is a bit more complicated

3. If two events do not share any outcomes, then the probability of being in either event is the sum of the probabilities of the individual events.

If we let A and B be events (subsets of the sample space) and we know that A and B are “disjoint” (have no common members), then the probability of an experiment giving an outcome in A or B is the probability of getting an outcome in A plus the probability of getting an outcome in B.