MA/CS 109 Lecture 12

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(sittin’ in for Prof Hall)
Today: Some examples

Recall, if we roll one die the sample space is $\{1,2,3,4,5,6\}$, while if we roll 2 dice, the sample space is
Using the Equally likely outcomes model, every square in this box (every outcome in the sample space) has the same probability, $1/36$.

In this case, events that depend only on the first die are independent from events that depend only on the roll of the second die.
Some questions are now easy to answer:

What is the probability of the two dice being the same?

6 outcomes both dice the same, so

\[ P(\text{both the same}) = \frac{6}{36} = \frac{1}{6} \]

What is the probability of the two dice being different?

\[ 1 - P(\text{both the same}) = 1 - \frac{1}{6} = \frac{5}{6} \]
If we roll a third die, then the computations become harder since we can’t visualize the sample space as easily...and computing probabilities and counting outcomes requires more cleverness...

If we roll three dice, then the sample space is 
\{(1,1,1),(1,1,2),\ldots,(1,2,1),\ldots,(6,6,6)\}

There are 6 outcomes for the first die, 6 for the second (so 6 x 6=36 for two dice)—then for each of the ways to roll the first two dice, there are 6 ways to roll the third. So there are (6 x 6) x 6=216 outcomes.
We can visualize the problem of 3 dice as a cube with 1 to 6 along each of the axes. Figuring the probability is the same as figuring a volume in this cube.

But if we have 4 dice, we can’t visualize a 4 dimensional box, so we need to generalize our counting technique (that is motivated by the geometry so far).
“Multiplication Principle”

The generalization is called the Multiplication Principle. It applies to a series of independent events.

If for each way of the N ways of doing a first thing there are M ways of doing a second thing, all giving a different result, then there are \((N \times M)\) ways of doing both things. With Q ways of doing a third thing, we have \((N \times M \times Q)\) ways of doing all three. And so on...

Here we have 36 ways of rolling 2 dice and for each of these rolls, 6 ways to roll the third, So \(6 \times 6 \times 6 = 216\) total outcomes.
Example: What is the probability of rolling three dice so that they are all the same number?

Well, there are 6 possible numbers, so 6 ways all three dice can have the same number, so

\[ P(\text{all three dice the same}) = \frac{6}{216} = \frac{1}{36}. \]

What is the probability of rolling three dice so that two or more are the same?

This turns out to be a more challenging problem...
We could make a list and count. (111), (1,1,2), (1,1,3), ... There are lots. Or we could be more clever.

First let’s recall that the probability of rolling three dice so that two or more are the same is equal to

$$1 - \text{(probability that all three dice are different)}$$

Turns out this is much easier to compute.
To compute the probability that all three dice are different, we count the outcomes in this event by thinking...

First roll the first die—it can be anything, so 6 possibilities.

Next roll the second die—it can be anything except the number of the first die, so for each number of the first die there are 5 possibilities.

So, $6 \times 5 = 30$ ways to roll two die and get two different numbers.
Now, roll the third die—for all three to be different, the third die can be anything except the numbers on the first two die, whatever they are. That means there are always 4 choices for the third die. Since there were 6 x 5 choices for the first two die to be different, that means there are 6 x 5 x 4 = 120 ways for all three to be different.

So the probability of rolling 3 dice and getting all three different is 120/216 = 0.56.
Back to the original problem -- what is the probability of rolling three dice and getting 2 or more numbers the same?

1- (120/216)= 96/216=0.44

Let’s do the same problem again...but with more excitement:
I’ll give each of you in this room $100 if there are 2 or more people in the room with the same “birthdate” that is, two or more people born on the same day of the year....

(I’m pretty sure I’m going to win!).
Why was this a good bet?

Really the same problem as before except with a die that has 365 sides (let’s ignore leap years for this model) and N rolls for N people in the class....

First, let’s compute the probability that everyone here has a different birthdate. One minus this is the probability that two or more people have the same birthdate.

The number of ways for the first person to have their birthdate is 365. The number of ways for the second person to have their birthdate so it is different is 364.
So the number of ways the first two people can have different birthdates is 365 x 364. But then there are 365 x 365 different pairs of birthdates, so the probability for two people to have different dates is \( \frac{365 \times 354}{365^2} = 0.9973 \). So probability that two people have same is 0.0027.

To keep going, the next person has 363 choices for a different birthdate, so 365 x 364 x 363 ways for three people to have different birthdates. This is \( \frac{365 \times 364 \times 363}{365^3} = 0.9918 \)

Probability of 3 having at least one shared birthdate: 0.0082

And so on. Each successive person has one less day available to have their birthdate if all are to be different. Let’s look at a spreadsheet.....
Probability that among $N$ people, there is at least one shared birthdate, as a function of $N$:

<table>
<thead>
<tr>
<th>$N$</th>
<th>Ways of All Birthdays</th>
<th>Ways of Diff Birthdays</th>
<th>$P(\text{all diff})$</th>
<th>$P(\text{not all diff})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>133225</td>
<td>132860</td>
<td>0.99726027</td>
<td>0.002739726</td>
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<tr>
<td>3</td>
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<td>48228180</td>
<td>0.99179583</td>
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<tr>
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<td>6.30256E+12</td>
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<tr>
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<tr>
<td>7</td>
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<td>8</td>
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<td>9</td>
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<tr>
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<tr>
<td>23</td>
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<td>4.22008E+58</td>
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<tr>
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<td>1.44327E+61</td>
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<tr>
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<td>4.92154E+63</td>
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<tr>
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<tr>
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<tr>
<td>100</td>
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<td>5.2093E+249</td>
<td>3.07249E-07</td>
<td>0.9999999693</td>
</tr>
</tbody>
</table>
Probability that among $N$ people, there is at least one shared birthdate, as a function of $N$: 

![Graph showing the probability that among $N$ people, there is at least one shared birthdate, as a function of $N$. The x-axis represents $N$ from 1 to 97, and the y-axis represents the probability from 0 to 1. The graph shows an increasing trend as $N$ increases, reaching nearly 1 by $N = 23$.](image-url)
The probability that all 100 of us have different birthdates is pretty small!! Only 0.0000003.

So the probability that I will win the bet (two or more people in the room will have the same birthdate) is

   \[ 1 - 0.0000003 = 0.9999997 \]

Since probability 1 is “certain”, this is a very good bet.
In fact, in this class (plus me), there are several multiple birthdays:

Two people share each of these birthdays:
1/19  2/15  3/19  4/1  5/23  5/31  10/15  10/30  
11/17  11/28  12/28

Three people share:  3/3

Four people share:  10/17