MA/CS 109 Lecture 4
Problem: Given a network (for example, the network that comes from the Konigsberg bridge problem) does it have an Euler circuit (path that crosses every edge exactly once and ends where it starts)?
Noticed (from examples)

If a network has an Euler circuit, if you imagine yourself standing at a node while a friend walks the Euler circuit, every time your friend arrives at your node by some edge (an “arriving” edge) there is a way for them to leave the node (by a “leaving” edge).

So at any node (that isn’t the starting node), the arriving nodes can be matched up with the leaving nodes.
Terminology

We say the “degree” of a node is the number of edges that touch the node.

The above says that if a network has an Euler circuit, then at every node that isn’t the starting node of the circuit, the number of arriving edges equals the number of leaving edges.
But every edge is used exactly once, so the degree of the node is

Number of arriving edges

+ Number of leaving edges

And since these numbers are equal...

Degree = 2 \times (Number of arriving edges).

That is—the degree must be an even number!!!
Starting Node

What about the starting node? If you are standing there, your friend starts the Euler circuit by leaving (on a “leaving” edge). Each time they return (on an “arriving” edge), the either leave again (on a “leaving” edge) like at other nodes

OR

the circuit is finished.
So, we can pair the arriving and leaving edges as before, by pairing the first leaving edge with the last arriving edge.

So the degree of the starting node is even also!!

Conjecture: If a network has an Euler circuit, then the degree of every node of the network must be an even number.
Proof:

Remember, a proof is an explanation of why a statement MUST be true. It isn’t evidence that it is true, OR an example where it is true.

Proofs are forever
Proof of our conjecture:

Suppose you have any network that has an Euler circuit (I’m not telling you ANYTHING else about the network other than it has an Euler circuit). At any node (that isn’t the starting node of the circuit), every edge touching the node is an arriving edge or a leaving edge of the circuit, and each arriving edge can be paired with the leaving edge that follows it in the circuit.
So the degree of the node is the Number of arriving edges + 
Number of leaving edges 
= 2(Number of arriving edges),
So it is even.

For the start node, the same is true if we pair the first edge (leaving) with the last edge (arriving) of the Euler circuit. So the degree of the start node is also even.

So every node has even degree. (QED..οεδ)
REMARKS:

1. This statement is true IN OUR MODEL WORLD. So, for example, remember we only allow strongly connected networks and we did no allow edges with both end points at the same node.
2. Be careful. This statement tells us about networks which have Euler circuits... it doesn’t tell us anything about networks that DON’T have Euler circuits.

In particular, if you have a network for which all the nodes have even degree, this theorem does NOT tell you it has an Euler circuit... it says “no comment” to that question. This is why you can trust mathematics—it says “no comment” when it doesn’t know for sure.
Did we solve the original problem?

Was there a way to walk around Konigsberg crossing each bridge exactly once, ending where you started?
To use our theorem (a statement that you know is true because it has a proof), we restate it one more time:

If a network has an Euler circuit then all the nodes have even degree.

Is the same as saying

If one or more of the nodes of a network has odd degree then the network does not have an Euler circuit.
What about Konigsberg?
Let’s look at the degrees of the nodes...
Node 1 has degree 3...We can stop. This network can NOT have an Euler circuit!
Reached the $$Fame$$ step...

Well Euler did OK—he made a good living as a “court” mathematician and his name is remembered. He is recognized as the founder of Network theory.
Modify the problem:

Could the people of Konigsberg have followed an Euler path? (A path that crosses every bridge exactly once—don’t care where you start or end?)

You looked at some examples in discussion...do you have a conjecture? Can you prove it???
Idea:

Suppose we have a network with an Euler path that is NOT an Euler circuit.

So this graph has a path that crosses every edge exactly once, but ends at a different node from the start.
Now...add one more edge that goes from the end node of the Euler path to the start node of the Euler path.

The resulting network has an Euler circuit!!

So we know that every node of the network with the added edge must have even degree. (We are sure of this!!)
Now take out the added edge to get back to the original network—the degree of every node must be even except for the start node and end nodes of the Euler path.

The start and end nodes must have odd degree because they are even with the added edge and they go down by one when we remove the edge.
So we have proven:

A network with an Euler path that is not an Euler circuit has exactly two nodes with odd degree and all other nodes must be even...

(Does that agree with your examples from discussion??)
Harder questions:

There is still more to do—

Given a network where every node has even degree, must it have an Euler circuit?

Given a network where all but two nodes have even degree, must it have an Euler path that is not an Euler network?...
We could talk about Network theory all semester (and facebook and maps and DNA...) and they may come back as they come up a lot in computer science (e.g., How does your phone find the “shortest path” between two addresses? Just trying all roads is way, way too hard...

But for now we’ll talk about one of the most powerful mathematics ideas—the first one you learned...Counting.