MA/CS 109 Lecture 6
Exponential functions

Exponential functions have “explosive growth”

Example: \( f(t) = 2^t \)

Growth in one time unit

\[
f(t+1) - f(t) = 2^{(t+1)} - 2^t = 2^t (2-1) = 2^t = f(t)
\]

So the rate of growth is proportional to the value of the function! Exponential growth has the property that the bigger the function, the faster it grows!
Consider the function $f(t) = 2^t$

Powers of 2:

- $f(1) = 2$
- $f(2) = 2^2 = 4$
- $f(3) = 2^3 = 8$
- $f(4) = 2^4 = 16$
- $f(5) = 2^5 = 32$
- $f(6) = 2^6 = 64$
- $f(7) = 2^7 = 128$
- $f(8) = 2^8 = 256$
- $f(9) = 2^9 = 512$
- $f(10) = 2^{10} = 1024$

Memorize!...will be very useful for computer science. In particular, $2^{10}$ is about 1000.
Exponential growth is faster than quadratic...

Blue linear(3t), orange quadratic (t²),
green exponential (2^t)
Exponential growth is a lot faster!!

Blue linear($3t$), orange quadratic ($t^2$), green exponential ($2^t$)
One more example of Exponential growth

Croissant have a long and interesting history (dating back to the legend of the bakers saving the city of Vienna from the Ottoman army attempting to tunnel under its walls in 1683)—the current version dates back to the homesick Marie Antoinette (an Austrian princess married off to Louis XVI of France)...
Recipe

Making a croissant involves creating layers of dough and butter. A layer of butter is incased between two layers of dough. The dough is then flattened and “folded” like a letter so that there are now 3 layers of butter separated by layers of dough.
Then the process is repeated...flattening and folding gives $3 \times 3 = 3^2 = 9$ layers...repeat gives $3 \times 3 \times 3 = 3^3 = 27$ layers, and so on. Each repeat (called a “turn”), multiplies the number of layers by 3—so the number of layers grows exponentially with the number of turns.
Sometimes, a “book fold” is used—this multiplies the number of layers by 4.

In any case, the number of layers grows quickly with the number of turns...but the thickness of the dough...

This points out the most important thing you will ever learn (maybe).
Most important thing you Ever Learn:
  Maybe
Exponential growth can not go on forever.
D’uh:

In a finite environment, unbounded growth cannot go on forever no matter what the rate of growth...
What is different about exponential growth is that you reach the limit of growth “suddenly”...

Examples:

- Limit 100. Linear growth 10t.
- Reaches limit at t=10
Quadrtratic growth:

Limit 100. Quadratic growth $t^2$.
Reaches limit at $t=10$
Exponential Growth

Limit 1000. Exponential growth $2^t$.
Reaches limit at $t=10$
All together:

Note how exponential growth both gets larger and growth rate accelerates as \( t \) gets larger!
So most of the growth takes place quickly at the end.
This is even more dramatic for larger times...

Same functions up to $t=20$, limit $1,000,000$
Note these are not much different for $t$ below 10 or 12 (on this scale...
This growth of the growth rate is a property of exponential growth, no matter what type—

Use of fossil fuels grows around 2% per year. So the more fossil fuels we use this year, the more growth for next year...but 2% isn’t much...right?

This means that the amount of fossil fuels used this year is

Amount last year +Amount last year \times 0.02

=Amount last year \times 1.02
So if $F(0)$ is used in year $0$
And $F(N)$ is amount used in year $N$

Amount used year $1 = F(1) = 1.02 \times F(0)$
Amount used year $2 = F(2) = 1.02 \times F(1) = 1.02 \times (1.02 \times F(0)) = 1.02^2 \times F(0)$

Amount used year $3 = F(3) = 1.02 \times F(2) = 1.02 \times (1.02^2 \times F(0)) = 1.02^3 \times F(0)$

And so on...
Doesn’t seem so bad...

2% is such as small amount of growth... But

\[1.02^{35}\] is approximately 2...

And this means that even with an exponential growth of only 2% per year, your size doubles every 35 years...

So 2% per year growth is the same as doubling every 35 years...
Still exponential growth...

Still the same shape
(In fact, doubles every 35 years).
So don’t get complacent

Ten years ago there was much talk of “peak oil”—Oil was running out, we were bound to use up all the “economically recoverable” oil soon!!!

But then “tar sands” started to be mined for oil—and Hydraulic Fracking was invented. Suddenly the amount of “economically recoverable oil” has doubled...we are all set.
For a while....if our rate of use keeps increasing at only 2% per year then our rate of use will double every 35 years.

So enough oil to last for 100 years at our current rate of use, will only actually last for 70 year if there is 2% per year growth in use (with any luck, within your lifetime!). Enough oil to last 200 years at our current rate, will only last 110 years if there is 2% per year growth in use.
Blue line is “constant rate of use”...
Red curve is 2% increase per year.
(red curve reaches 100 at about 70 and 200 at about 110).
Even worse... exponential growth is dangerous because there isn’t much time between abundance and shortage...the end comes quickly.

There are many examples of this in economics — ”economic bubble” followed by a “crash” (pop?)

Tulip mania—in 1634-37 the price of tulip bulbs doubled 8 times. But exponential growth can not go on forever...by 1638 the price had fallen to below 1634 prices—a “crash”.
Housing prices 2000-2006...then crash...

There are many other examples of markets that grew exponentially—then stopped growing exponentially with a “correction” or a “crash”.

DON’T PANIC…Exponential growth doesn’t have to lead to a crash—when email was first invented, the number of email servers grew exponentially for many years but then the rate of growth slowed. It is still growing, just not as fast as before.
Even when there is a crash—the world doesn’t usually end...(hasn’t ended any day of my life so far...)

But” exponential growth can’t go on forever” implies that something must change. When radical change happens, it is always good for some and bad for others...sometimes very good for a few and very bad for many.

Moral: When you see something growing exponentially, don’t expect it to last forever and expect change.
One more type of function...

There are functions that grow even faster than exponentially... for example

if $f(t) = 2^t$ then $s(t) = 2^{f(t)}$ or $2$ to the $2$ to the $t$ grows even faster!

There are functions that grow more slowly even than linear functions...
One more type of function:

Question: How many digits does it take to represent a number?
One more type of function:

Question: How many digits does it take to represent a number?

0-9  one digit
10-99 two digits
100-999 three digits
1000-9999 four digits
So the function $D(N) =$ number of digits it takes to represent $N$, grows very very slowly.

Even $D(1,000,000,000) = 10$

What kind of functions grow this slowly?

What are they good for?
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Logrhythms !!
Discussion section—Prof. Snyder’s family

In this family each generation is twice the size of the previous generation and at holiday season each generation gets together and gives one substantial gift to a charity in the name of the family.

The oldest generation had 2 members—1 gift. The next generation had 4 members (so a total of 6 members of the family)—2 gifts
The next generation has 8 members for a total of 14 members—and 3 gifts
The next generation has 16 members for a total of 30 members—and 4 gifts

The number of gifts is growing very slowly—only as fast as the number of generations (which is much less than the number of people in the family).
This is definitely a different sort of growth. Here the rate of growth decreases as the numbers get bigger.
Log functions

The type of function described above is called a Logarithm function... We will only use one type of Log function—Log base 2, denoted $\log_2(N)$.

Definition: $\log_2(N)$ is the number you raise 2 to in order to obtain $N$.

So $\log_2(4) = 2$ because $2^2 = 4$

$\log_2(8) = 3$ because $2^3 = 8$

And so on...
So $\log_2(64) = 6$ because $2^6 = 64$

And

$\log_2(1000)$ is approximately 10 because $2^{10}$ is about 1000.

$\log_2(2000)$ is about 11 because $2^{11}$ is about 2000