MA/CS 109 Lecture 7
Log functions

The type of function described above is called a Logarithm function...We will only use one type of Log function—Log base 2, denoted \( \log_2(N) \).

Definition: \( \log_2(N) \) is the number you raise 2 to in order to obtain N.

So \( \log_2(4) = 2 \) because \( 2^2 = 4 \)

\( \log_2(8) = 3 \) because \( 2^3 = 8 \)

And so on...
So \( \log_2(64) = 6 \) because \( 2^6 = 64 \)

And

\( \log_2(1000) \) is approximately \( 10 \) because \( 2^{10} \) is about 1000.

\( \log_2(2000) \) is about 11 because \( 2^{11} \) is about 2000

\( \log_2(100) \) is

between \( \log_2(64) = 6 \) (because \( 64 = 2^6 \))

and \( \log_2(128) = 7 \) (because \( 128 = 2^7 \))
How to compute $\log_2(N)$

In Excel use

= Log(N,2)

Means”” Log base 2 of N””
So type
=Log(100,2)

The number 6.6438 appears in the square...
(Note, it is between 6 and 7 as predicted!)
Property of Logs

A nice property of $\log_2$ we will use is that $\log_2$ grows so slowly, that it can turn exponential growth into linear growth...

This is a property of Log’s that you may remember from study for the SAT’s

$$\log_2(a^t) = t \log_2(a)$$

That is Log’s turn exponentiation in multiplication.
Suppose we have data...

<table>
<thead>
<tr>
<th>t</th>
<th>F(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.7</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>10</td>
<td>202</td>
</tr>
<tr>
<td>15</td>
<td>2862</td>
</tr>
<tr>
<td>20</td>
<td>40642</td>
</tr>
<tr>
<td>25</td>
<td>577062</td>
</tr>
<tr>
<td>30</td>
<td>8193465</td>
</tr>
</tbody>
</table>
Graph looks like exponential growth...
But if we take the $\log_2$ of the data

<table>
<thead>
<tr>
<th>$t$</th>
<th>$F(t)$</th>
<th>$\log_2(F(t))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.7</td>
<td>0.76</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>3.83</td>
</tr>
<tr>
<td>10</td>
<td>202</td>
<td>7.65</td>
</tr>
<tr>
<td>15</td>
<td>2862</td>
<td>11.48</td>
</tr>
<tr>
<td>20</td>
<td>40642</td>
<td>15.31</td>
</tr>
<tr>
<td>25</td>
<td>577062</td>
<td>19.14</td>
</tr>
<tr>
<td>30</td>
<td>8193465</td>
<td>22.97</td>
</tr>
</tbody>
</table>
Graphing the Log$_2$ of the data the exponential growth becomes linear growth...

So Log$_2$ turns exponential growth into linear growth.
Graphing the $\log_2$ of the data the exponential growth becomes linear growth...

So $\log_2$ turns exponential growth in to linear growth.
Log graphs

You see this kind of graph all the time...look at the numbers on the vertical axis. If they grow non-uniformly, then you expect this is the graph of the Log of the data.
Growth of Massachusetts and North Carolina populations

The graphs below are the populations of Massachusetts and North Carolina from 1800 to 2000 (U.S. Census figures). (Blue Mass, Orange NC.)
Both populations grow a great deal from about 450,000 in 1800 to 6.3 million for Massachusetts and 8 million for North Carolina...is this exponential growth??

To find out, we take \( \log_2 \) of the populations (in millions) and plot these number

For example \( \log_2(0.45)\approx -1.15 \),
\( \log_2(6)\approx 2.58 \),
\( \log_2(8)\approx 3 \), and so on.
Recall Logs turn exponential growth into linear growth...so if the populations are growing exponentially, we expect the Log of the population to give a line...for Massachusetts...

Not very much like a line...turns down. Not exponential growth.
But North Carolina

The graph of $\log_2$ of the population is a line! So the population of North Carolina is growing exponentially! (Buy land in North Carolina...)
To tell if a function is growing exponentially (or faster) take the $\log_2$ of its values and see if this is growing linearly (or faster).

OK...We have a zoo of functions. We have looked at these functions in terms of their growth rates—on the main event, modeling populations.
A Mathematical Model in Population Ecology... (Invasive species)
Population Models

We want to do one more problem in “Applied” math...There are many, many examples we could choose, but we want to choose one in which we can “start from scratch”.

Choose “mathematical ecology” and look at simple population models.
Template for Doing Mathematics

Problem

| Model-----------------Repeat---------------------------|
| Modify

Examples/Conjectures

| Model

Proof---Did we answer the question?---No

Yes—Fame $$$
First Example Model

As our first example, we take a the population of rabbits in Australia. Rabbits were introduced for food and sport in the late 1700’s but only spread rapidly after a release in 1859.

There were few predators, few diseases and food was available, so the rabbits started to reproduce...well, like rabbits!
Problem:
Predict the future population of an (invasive or new) species (like rabbits in Australia) with few predators and plenty of food.

Model: Biology(!)

Assumptions (This is the “model building” step of our template. We use (very) simple biology).
Assumptions:

1. The number of rabbits at the start of next year is the number of rabbits at the start of this year minus the number that die plus the number that are born.

2. The number of rabbits that die in a given year is proportional to the number of rabbits alive at the start of the year.

3. The number of rabbits born in a given year is proportional to the number of rabbits alive at the start of the year.
To make this useful we have to write it “precisely” in a way that we can make computations...

First, some notation...our assumptions refer to the “number of rabbits alive at the start of a year”—this is a long phrase. Let’s use “rabbits this year” for “number of rabbits alive at the start of this year” and “rabbits next year” for “number of rabbits alive at the start of next year”

We will also use the equals sign “=” for the verb “is” as well as “+” for “plus” and “-” for minus.
Assumption 1 says:
Rabbits next year = rabbits this year – rabbits that die this year + rabbits that are born this year.

Assumption 2 says;
Rabbits that die this year “is proportional to” the number that are alive at the start of the year (“rabbits this year”)
“Proportional to “ means “a constant times”

So Assumption 2 says
Rabbits that die this year =
constant x (rabbits this year)
Let’s call this constant d for “death rate”
constant.

So
Rabbits that die this year = d x rabbits this year
Assumption 3 says sort of the same thing:
Rabbits born this year is proportional to rabbits this year or

Rabbits born this year = b x rabbits this year

where b is the “birth rate” constant.
Putting this together, we get
Rabbits next year
   =Rabbits this year – d x rabbits this year
      + b x rabbits this year

That is
Rabbits next year = (1 – d + b) Rabbits this year

If d is bigger than b, then 1-d+b<1 and the rabbits next year is only a fraction of the rabbits this year.
But, if $b$ is larger than $d$ (birth rate bigger than death rate) then

$$(1 - d + b) > 1$$

And the rabbits next year is larger by the factor $(1 - d + b)$ than the rabbits this year. Let’s write this number more efficiently.

Let $k = (1 - d + b)$ and call this number the “Growth rate constant”.
Model

Our model is

Rabbit next year = k x Rabbits this year.

This is still kind of clunky... Let’s get better notation. Let

R(t) = number of rabbits at the start of year t

So R(0) = number of rabbits at the start of year 0 (say the year rabbits were first released).
So

\[ R(1) = \text{number of rabbits at the start of year 1} \]
\[ R(2) = \text{number of rabbits at the start of year 2} \]

And so on...

If this is year \( t \) then “rabbits this year” = \( R(t) \)
and “rabbits next year” = \( R(t+1) \)
Model

Our model, with all this notation, is

\[ R(t+1) = k \ R(t) \]

(in words, rabbits next year is \( k \) times rabbits this year).

We are assuming the population grows, so we are assuming \( k>1 \).
What does the model predict?

The model says that if we know $R(0)$ (rabbits at year zero) then we can figure out $R(1)$

$$R(1) = kR(0)$$

(provided we know $k$. $R(0)$ is determined by how many rabbits are released, $k$ is determined by rabbit biology and the environment.)

Knowing $R(1)$, we can predict $R(2)$

$$R(2) = kR(1)$$
But wait,
\[ R(1) = kR(0) \text{ and } R(2) = kR(1) \]

Gives
\[ R(2) = k (k(R(0))) = k^2 R(0) \]

Similarly
\[ R(3) = kR(2) \]

So
\[ R(3) = kR(2) = k (k^2 R(0)) = k^3 R(0) \]
In general, if
\[ R(t) = k^t R(0) \]
Then
\[ R(t+1) = kR(t) = k (k^t R(0)) = k^{(t+1)} R(0). \]

So, we have established that
\[ R(t) = k^t R(0) \]
For all \( t = 1, 2, 3, \ldots \) forever.
(This is an example of a “proof by induction”—Show the pattern holds at the start and show it keeps holding, then it must always hold.)