MA/CS 109 Lecture 8
Last time

Exponential Growth Model

“Rabbits Next year equals growth rate constant times rabbits this year”

Or in notation

\[ R(t+1) = k \ R(t) \]

\( R(t) \) = rabbits at the start of year \( t \)

\( k \) = growth rate constant (\( k>1 \) for growth)
What does the model predict?

The model says that if we know $R(0)$ (rabbits at year zero) then we can figure out $R(1)$

\[ R(1) = kR(0) \]

(provided we know $k$. $R(0)$ is determined by how many rabbits are released, $k$ is determined by rabbit biology and the environment.)

Knowing $R(1)$, we can predict $R(2)$

\[ R(2) = kR(1) \]
But wait,

\[ R(1) = kR(0) \text{ and } R(2) = kR(1) \]

Gives

\[ R(2) = k(k(R(0))) = k^2 R(0) \]

Similarly

\[ R(3) = kR(2) \]

So

\[ R(3) = kR(2) = k(k^2 R(0)) = k^3 R(0) \]
In general, if
\[ R(t) = k^t R(0) \]
Then
\[ R(t+1) = kR(t) = k (k^t R(0)) = k^{(t+1)} R(0). \]

So, we have established that
\[ R(t) = k^t R(0) \]
For all \( t = 1,2,3,\ldots \) forever.
(This is an example of a “proof by induction”—Show the pattern holds at the start and show it keeps holding, then it must always hold.)
Predictions of the Model

So what does our model predict? Because we are so lucky as to be able to write a formula for the rabbits in the $t^{th}$ year. We see that we

$$R(t) = k^t R(0)$$

This is exponential growth (because $k>1$)!
Example

Let’s assume \( R(0)=1 \) and the growth rate constant is \( k=2 \) (OK, the biology is suspicious... but this is just an example).

Then

\[
R(t) = 2^t \quad R(0) = 2^t
\]

This is a function we are familiar with!
Graphing we see

Exponential Growth!
What do we learn?

In the simplest model, the invasive species will grow exponentially (all we need is $R(0)>0$ and $k>1$). Remember $k=1-d+b$ so need birth rate more than death rate.

What else…?

Well, suppose instead of just 1 rabbit, we start with a more realistic $R(0)=2$ rabbits. With this initial population we have that

$$R(t) = 2^t R(0) = 2^t(2)$$
The blue is $R(0)=1$ and orange $R(0)=2$. The orange is always twice the size of the blue. The difference grows exponentially! Small error at $t=0$ grow exponentially!!
Important observation

Model says if we make a small error in the population at the start, after a relatively short time, prediction of the model will be very, very different.

“Error grows exponentially”

(In our example, both predictions are “lots of rabbits”. But we will see that this will be the basis of “Chaos Theory”!!)
Bad News...

This is pretty bad news—it says that eventually we will be swimming in rabbits -- see the old Star Trek episode “The trouble with tribbles” at https://www.youtube.com/watch?v=WXQ0CMp6Bi8.

This is not dissimilar to what happens when an infectious disease infects your body. Its population grows exponentially until it is so overwhelming that you become sick.
The Exponential Growth Model is exceptionally nice...we can give a nice formula for the predicted population as many years in the future as we like...

But even when we have formulas, graphs are sometimes better for explaining what is going on...

There are TWO ways to graph the Exponential Growth Model—each with its own advantages.
The first type of graph shows the predicted population way into the future for one choice of initial population \( R(0) \).
The graph of this prediction is a “time series” and for the Exponential Growth model, typically looks like this...
Another way to graph the model is to make a graph with “rabbits this year” on the horizontal axis and “rabbits next year” on the vertical axis.

This allows us to make only a one year prediction into the future, but we can make this prediction for any population of rabbits this year.
Horizontal is rabbits this year, vertical is rabbits next year. Using this graph we can predict one year in the future for any number of rabbits this year...Here for growth rate constant equal to 2.
We can add the orange line

Rabbits this year = Rabbits next year

for comparison. We see that with k=2, the population next year is always larger than the population this year, except if Rabbits this year = 0 ("Extinct is forever").
Also, we see that as the population this year gets larger, the growth in population over one year gets greater.

That is, larger population means faster growth! This is the calling card of the Exponential Growth Model.
A smaller $k$ value means (here $k=1.4$) the population next year is not as much bigger than the population this year.

Note that no matter what $k>1$ value we have, the hallmark of exponential growth is visible in this picture—the bigger the population this year, the more it grows by next year (bigger it is, faster it grows...
Recap: Exponential model predictions

We showed

\[ R(t) = k^t R(0) \]

which is an exponential function (variable \( t \) is in the exponent).

Since \( k>1 \), this model predicts exponential growth...
Larger growth rate constant \( k \) gives faster growth, but always have the distinctive shape.

In both graphs \( R(0)=1 \). On the left \( k=1.4 \), on the right \( k=2 \). Note difference in vertical scale, but the shape is the same.
And we can represent the model graphically two ways:

Time series--choose R(0), plot population far into the future.
Or, Predict one year in the future for any population.

The model
\[ R(t+1) = kR(t) \]
is a linear relationship between \( R(t) \) and \( R(t+1) \) with slope \( k \).
(Blue is the model, \( k=2 \), Orange is population this year = population next year for reference)

For any population this year, find it on the horizontal axis and go up to the blue line and over to get the population next year.
We do see exponential growth in some cases of invasive species!

But exponential growth can’t go on forever… Australia isn’t swimming in rabbits.

So for long-term predictions we need to build in some limits to growth.

First limit to growth: Suppose we try to control the invasive species by “Harvesting”, that is, we “remove” members of the invasive species, trying to control or even wipe out the species.
Exponential Growth with Harvesting Model

“Harvesting” costs money, so usually we can’t just go out and remove every member of the invasive creature... More typically, some amount of money is allocated for harvesting and that can remove some fixed number of individuals.

Assumption 4: (added to the above) A certain number of the population will be removed by harvesting each year.
Build in the new assumption to the model.

Rabbits are born and die the same as before, the modification for the new assumption is that a certain number are harvested...
Build in the new assumption to the model.

Rabbits are born and die the same as before, but we add the new assumption:

Assumption 4: Each year, H rabbits are harvested (or otherwise removed) from the population.
Build in the new assumption to the model.

Old model
Rabbits next year = growth rate constant \times Rabbits this year

New model
Rabbits next year = growth rate constant \times Rabbits this year
- Harvested rabbits
In our notation
Rabbits next year
= growth rate x rabbits this year - harvested

Becomes

$$R(t+1) = k R(t) - H$$

where $H$ is the "harvesting rate", the number removed each year.
What does this model predict?

Example: Take $k=2$ again and let $R(0)=1$ and let $H=4$ then

$$R(1) = 2 \times 1 - 4 = 2 \times 4 = -2$$

Since we can’t have negative rabbits, we interpret this as “rabbits have gone extinct”

This is very different! In this case harvesting has wiped out the rabbits.
What if \( k=2, \ H=4 \) as above, but \( R(0)=5 \)?

Then next year

\[
R(1) = 2 \times R(0) - 4 = 2 \times 5 - 4 = 10 - 4 = 6
\]

So the population in year 1 is now 6 rabbits. The population went up a little even with the harvesting.
This is very different than the exponential growth model. For that exponential growth with $k=2$, any positive initial value of $R(0)$ gave a rapidly growing population.

For exponential growth with harvesting, we see that some initial populations can be wiped out by harvesting, but larger initial populations still grow...

How can we “see” how this new model works?
Recall: Graph of the Exponential model

The model

\[ R(t+1) = kR(t) \]

is a linear relationship between \( R(t) \) and \( R(t+1) \) with slope \( k \).
(Blue is the model, \( k=2 \), Orange is population this year = population next year for reference)

For any population this year, find it on the horizontal axis and go up to the blue line and over to get the population next year.
If we draw the same type of graph for the model with harvesting, we get a different picture.

Rabbits next year

Here $k=2$, $H=5$. Subtracting 5 pushes the blue model graph down 5 units.
What does this predict??

If the population this year is small (4 or less) then the population next year is less than the population this year—That is good! We are winning!
In fact, if the population is small enough (2 or less) then the population next year is negative—so the rabbits are extinct.

So, we start harvesting when the population is small enough (4 or less), the population will decrease and be forced into extinction!

WE WIN.
But, if the population this year is large (5 or bigger) then the population next year is larger than the population this year. And, the bigger the population, the more it grows.

So large population grows exponentially!
Moral

Harvesting can work to eliminate an invasive species IF the harvesting rate $H$ is large enough AND you start when the population is small enough.

(The sooner you start, the smaller the amount of harvesting will drive the population to extinction.)

Why, in California, pesticide spraying is so aggressive when the Mediterranean fruit fly is detected.
One more observation. Between the population value that leads to decreasing population and increasing population there is a value of the population where population next year equals population this year. No change. This is a “fixed point”. (Where blue and orange cross.)
Discussion next week will look at this model applied to an ecological comedy (or tragedy...)

This slightly more complicated model has considerably more variability in its predictions depending on the details like the value of $R(0)$. Starting to get enough structure in our model to begin to see the more realistic variety of behaviors in nature.
Criticism of this model

There are several weaknesses of this model.

Most important is the lack of “stability”. In nature we see the population of the invasive species grow, but then level out and stay fixed. In this model, if we are at the fixed point, one extra rabbit will cause an explosion of the population, on fewer and the population dies out.