1 Tools for predicting the future

Can we predict the future?

In general, the answer is “no”. If I could predict the future, I would have already made a fortune in the stock market or the lottery and be living on a private island...

However, there are some situations in which the future can be at least approximated. For example, if a tiny spot of mold starts to grow on a piece of bread, it is not hard to predict that the spot will grow larger. If a few rabbits are left on an island which has fresh water, vegetation and no predators, it is likely that the number of rabbits will grow fairly quickly. If some foxes are imported to the island, then it is likely that the rabbit population will decline.

These aren’t particularly astounding predictions. We can make more interesting predictions by making them more precise, i.e., more quantitative. In order to do so, we need a mathematical tool that you already know about. What we need is a way to relate the value of one quantity to the value of another quantity, e.g., the population of rabbits to the date in the future. That is, a way to express the population as a “function” of time.

The most amazing thing about functions is their adaptability. The same function can come up describing the number of steps in an algorithm, the number of combinations of friends in a class and the number of square feet in a room. Sometimes this adaptability gives new ways to think about old problems.

1.1 Functions

You have seen many functions in your mathematics training. In its most elemental form, a function is a way to relate two quantities. For example, in the discussion above, we relate the time (measured in months or years) with the population of rabbits on an island. Given the date, the function gives back a population.

Functions can be specified in many ways. We can use data (obtained by counting rabbits on the island directly) to give a table of values. We can show a picture of the function as a graph, with the time along the horizontal axis and the number of rabbits as the vertical axis. Finally, we can sometimes write a formula for a function which gives a way to explicitly compute the population from the time.

In mathematics classes, functions usually appear as formulas, but tables and graphs are much more common in everyday life. We should try to think of functions in all of these manifestations.

1.2 A Zoo of Functions

For the next two lectures (or so), we will build examples of functions—remembering to look at all the representations (graphs, tables and formulas). Many of our examples of functions
will also be related to “stories”. The goal here is to give yet another way to think about the properties of the function. As a “hidden agenda”, a number of the stories will involve setting up the functions by counting something. We will do a lot of counting during the semester and the techniques of counting we use here will reappear frequently.

Instead of thinking of this as a list of examples, think of it as a “zoo”. You visit a zoo to get a look at different animals you do not usually see and better appreciate properties that you might already know. While we know from TV that a tiger is strong and a giraffe is tall, actually seeing them in real life gives new insight into these qualities.

1.3 Growth Rates and Polynomials

We focus on the “growth rate” of different types of functions. That is, how quickly (or slowly) the output number of the function increases as the input increases. This is clearly important for population data—we want to know how quickly the population of rabbits is going to grow. Other qualitative features (like the “wiggliness” of a graph or even its value at particular points) can be important too, depending on the function and what we are trying to predict.

LINEAR FUNCTIONS:

<table>
<thead>
<tr>
<th>Algebra Formulas</th>
<th>Graph XY AXIS</th>
<th>Story Made up</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linear functions</td>
<td><img src="image1" alt="Graph for L(N) = N" /></td>
<td>Given a room of N people, how many ways are there to choose one?</td>
</tr>
<tr>
<td>$L(N) = N$</td>
<td><img src="image2" alt="Graph for L(N) = N" /></td>
<td></td>
</tr>
<tr>
<td>$P(N) = 8.000 + .5N$</td>
<td><img src="image3" alt="Graph for P(N) = 8.000 + .5N" /></td>
<td>You start a job at $8000/year and get $500/year raise. What are you paid in year N?</td>
</tr>
</tbody>
</table>
The linear functions are among the “tamest” and most familiar in our zoo – the cat and
dog of the group. Remember that a line can be defined by its slope (how steep it is) and
its vertical intercept (where it crosses the vertical axis). The slope says how fast the line is
increasing (positive slope) or decreasing (negative slope). So \( L(N) \) has a steeper slope but a
lower \( y \)-intercept than \( P(N) \). We should note here that even though \( L(N) \) eventually grows
faster than \( P(N) \), we can compare them both on the same graph even for large \( N \) (\( \approx 30 \)).

QUADRATIC FUNCTIONS

Suppose you have a class of students and you want to pick out two of them. How many
ways are there to choose two students? Of course, it depends on how many students are
in the class, so let this number be \( N \). To figure out the exact number of ways there are
of choosing two students we first need to be more specific. (This is an example of “model
building”–being precise so that we know what we are talking about.)

- Can a student be chosen twice, or do we want to make sure we pick two different
  students?
- Does the order of choice matter (e.g., are we picking a president and a vice-president),
or does order of choice not matter (e.g., are we picking a two person committee)?

Depending on the answers, we get different equations. Both the similarities and the differ-
ences in the possible situations are interesting.

First let’s suppose that we can choose the same person twice and that the order of choice
does matter, e.g., we are giving away two different door prizes and the same person can win
both.

To count the number of ways to choose two people in this case, we think of first choosing
the first person, then choosing the second. There are \( N \) ways of choosing the first person.
For each of the choice of first person there are \( N \) ways to choose the second.

We could count the possibilities by giving the people names (say \( A,B,C, \ldots \) ) then making
a list of first choice, then second choice: \( AA,AB,AC, \ldots, BA, BB, \ldots \), and count the list.
This would be painfully tedious. We can organize the list better by making a table:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>...</th>
<th>Nth person</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>AA</td>
<td>BA</td>
<td>CA</td>
<td>...</td>
<td>Nth personA</td>
</tr>
<tr>
<td>B</td>
<td>AB</td>
<td>BB</td>
<td>CB</td>
<td>...</td>
<td>Nth personB</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nth person</td>
<td>ANth person</td>
<td>BNth person</td>
<td>...</td>
<td>Nth person Nth Person</td>
<td></td>
</tr>
</tbody>
</table>
each column there are the $N$ possible choices of second person. Just like finding the area of a rectangle, to find the total number of possible choices, we multiply the number of columns times the number of rows.

This works because for each choice of the first person there are the same number of choices for the second person. In this case there are $N$ ways of choosing each person to the total is $N \cdot N = N^2$. This technique of counting is called the “multiplication principle” and we will see the idea applied frequently to simplify the counting process.

The resulting function is $Q(N) = N^2$ and is pictured below. Also in the same graph picture below is the graph of the a linear function. The graph of the linear function $L(N)$ from the previous example is also on this graph. Note how much larger the quadratic function is than the linear function for large $N$. We can barely see the growth in the linear function crushed against the horizontal axis compared to the quadratic function. The linear function is the same as before—but the scale of the picture is different.

For the next example, we change the problem above. Suppose we want to choose two people from a group of $N$ people, but we can not pick the same person twice. We still assume that the order of choice matters (we want a president and vice-president).

To count the number of possibilities, we can again use the multiplication principle if we are careful. There are $N$ ways to choose the first person. For any choice of first person there are now $N - 1$ ways to choose the second person because the first person chosen has to be left out of the list of possible second choices. Notice that while the list of possible second choices is slightly different for each choice of first person, the number of possible choices is the same for each possible choice of the first person. So we can use the multiplication principle again and see that the total number of possible choices is $R(N) = N \cdot (N - 1) = N^2 - N$.

We do one more example of a quadratic function. Suppose we are choosing two people from a group of $N$ people where order does not matter (i.e., choosing AB is considered the same as choosing BA), but we can not pick the same person twice. We start by counting the ways to choose two people where order matters, or $R(N) = N^2 - N$, but then we note that every possibility has been counted exactly twice. So to eliminate the double counting, we divide by 2,

$$H(N) = \frac{N^2 - N}{2}.$$ 

In the table above, this corresponds to counting all the entries above the diagonal from upper right to lower left—those entries below the diagonal are just repeats of those above in the other order.

**CUBIC FUNCTIONS**

In the graph of $H(N)$ below we also show the graph of $R(N)$ and a linear function. Note that while $R(N)$ is larger than $H(N)$ for large $N$, we can see that both functions are growing at a comparable rate. The linear function, on the other hand, seems to be barely growing at all. All three of the functions $Q(N)$, $R(N)$ and $H(N)$ are said to be growing “quadratically”
and quadratic growth is always eventually larger than linear growth.

The same ideas continue on to the case of choosing three people from \( N \). For example, if order of choice matters and you can pick the same person twice or even three times then there are \( N \) ways to pick the first person, \( N \) ways to pick the second person and \( N \) ways to pick the third person. So (multiplication principle) the total number of ways is \( C(N) = N \cdot N \cdot N = N^3 \).

If order matters, but being chosen more than once is not allowed, then there are \( N \) possible first choices, for each first choice there remain \( N - 1 \) possible second choices and once these two choices have been made there are \( N - 2 \) third choices. So (multiplication principle) the total number of possible choices is

\[
K(N) = N \cdot (N - 1) \cdot (N - 2) = N^3 - 3N^2 + 2N.
\]

Both \( C(N) \) and \( K(N) \) “grow cubically”.

The graph of \( K(N) \) above also shows \( C(N) \) (which is larger than \( K(N) \)) and a function that grows quadratically. Moral: Cubically growing functions are eventually (for large \( N \)) much larger than quadratically growing function which are eventually larger than linearly growing functions. We can see that a hierarchy of functions is developing depending on which grows faster.

If the largest term is \( N \) raised to some power, then we say that the function “grows polynomially” or has “polynomial growth”. This covers a wide range of functions. We all know that \( N^{10} \) is big when \( N \) is big, but \( N^{20} \) is much much bigger. Surprisingly, there are commonly occurring functions that grow even faster than \( N \) to any power.

### 1.4 Exponential Growth

Exponential functions are extremely important because they frequently involve money (and everybody cares about money). For example, suppose you borrow 1 dollar at an interest rate of 50 percent per day. After one day, you owe the original dollar, plus interest, or

\[
1 + 0.5 \cdot 1 = 1.5 \cdot 1 = 1.5.
\]

After two days, you owe the original dollar, the interest for the first day, the interest for the second day AND the interest on the interest from the first day, or

\[
1 + 0.5 \cdot 1 + 0.5 \cdot 1 + 0.5 \cdot (0.5 \cdot 1) = 1.5 \cdot 1.5 \cdot 1 = 1.5^2 \cdot 1.
\]

It is this compounding, the interest on the interest, that becomes, after a relatively short time, the most significant factor. After \( N \) days, you will owe \( P(N) = 1.5^N \) dollars. As you may know from your credit card bill, this sort of function can grow very, very fast.

In the graph below, you see the function \( P(N) \) along with a quadratically growing \( Q(N) \). Note how quickly the exponential function “takes off”—its growth rate for large \( N \) is very much greater than a quadratic function (or any function with polynomial growth).
<table>
<thead>
<tr>
<th>Algebra Formulas</th>
<th>Graph XY AXIS</th>
<th>Story Made up</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quadratic functions</strong></td>
<td><img src="image1.png" alt="Graph" /></td>
<td>From a bag of $N$ prizes, how many ways are there to pick 2 with replacement? (order matters)</td>
</tr>
<tr>
<td>$Q(N) = N \cdot N = N^2$</td>
<td><img src="image2.png" alt="Graph" /></td>
<td></td>
</tr>
<tr>
<td>$R(N) = N \cdot (N - 1) = N^2 - N$</td>
<td><img src="image3.png" alt="Graph" /></td>
<td>Number of ways of picking two people from a room on $N$ people where the order of choice matters.</td>
</tr>
<tr>
<td>$H(N) = \frac{N^2 - N}{2}$</td>
<td><img src="image4.png" alt="Graph" /></td>
<td>Number of ways of picking two people from a room of $N$ people where order of choice does not matter.</td>
</tr>
<tr>
<td><strong>Cubic functions</strong></td>
<td><img src="image5.png" alt="Graph" /></td>
<td>Choose 3 people with replacement, order matters.</td>
</tr>
<tr>
<td>$C(N) = N \cdot N \cdot N = N^3$</td>
<td><img src="image6.png" alt="Graph" /></td>
<td>Choose 3 people, order matters, but no replacement.</td>
</tr>
<tr>
<td>$K(N) = N \cdot (N - 1) \cdot (N - 2) = N^3 - 3N^2 + 2N$</td>
<td><img src="image7.png" alt="Graph" /></td>
<td></td>
</tr>
</tbody>
</table>

6
<table>
<thead>
<tr>
<th>Algebra Formulas</th>
<th>Graph XY AXIS</th>
<th>Story Made up</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential functions</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P(N) = 1.5^N$</td>
<td></td>
<td>Borrow 1 dollar at 50 percent interest per day. After one day you owe 1.50, after two days you owe $1.5 \cdot 1.5 = 2.25$ (don’t forget the interest on the interest!), after three days you owe $1.5 \cdot 2.25 = 3.375$, <em>ldots</em></td>
</tr>
<tr>
<td>$S(N) = 2^N$</td>
<td></td>
<td>You bet double or nothing over and over, and keep winning!</td>
</tr>
<tr>
<td>Factorials</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$F(N)$</td>
<td></td>
<td>How many ways are there to put $N$ people in order? ($N$ ways to choose first, $N - 1$ to choose second, etc.)</td>
</tr>
<tr>
<td>$= N!$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$= N \cdot (N - 1) \cdots 2 \cdot 1$ (N factorial)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Even bigger</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E(N) = N^N$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$= \underbrace{N \cdot N \cdots N}_N$</td>
<td></td>
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</tbody>
</table>

7
As another example of exponential growth, consider the following counting problem. Suppose we have a room of \( N \) people and we want to divide them into two groups—say group One and group Two.

A simplification we can do to make counting easier is to notice that we only have to count the number of ways of making up group One. If everybody that isn’t in group One is automatically in group Two, then knowing who is in group One is enough information. Now, we could, with sufficient patience, make a list of all possible ways to subdivide the people. However, even if there are only 3 people, A, B and C, the list of possibilities is quite large. We could have nobody in group One, we could have just A, just B or just C in group One, or just AB, AC, or BC are in group One or all three ABC are in group One. We already have eight ways to choose group One from only 3 people!

We take a slightly different approach to this counting problem. Instead of “choosing” people for group One, we think of going down the entire list of \( N \) people and assigning each person to either group One or not (i.e., group One or group Two). So we line the people up, and for each person in turn, we have two choices, group One or not. If we have just one person, then there are only two ways to choose group One (put the person in or not). For two people, we have two possible choices for the first person (group One or not) and two choices for the second person (group One or not) and the choices do not effect each other. Hence there are \( 2 \cdot 2 = 4 \) ways to choose. This is the multiplication principle again.

For each additional person, we have another two possible choices (group One or not) for that person, so the number of possible membership lists for group One doubles with each additional person. Hence, for \( N \) people, the total number of ways of assignment to group One is \( S(N) = 2^N \).

Note that the argument above is an example of induction. The key step is to note that no matter how many people we start with, for each additional person, we double the number of possibilities for group One because the new person will either be in group One or not. The “common language” way of describing this is to show some examples, “two people gives 4 choices, three people gives 8 choices”, then say, in a loud voice, “see?” and leave it to the listener to identify why the pattern holds. This is just rude. To be complete (i.e., to give a proof rather than just examples), you should say why the pattern of doubling for each additional person holds no matter how many people there are.

### 1.5 Faster growth

There are functions that grow even faster than exponential functions. For example, suppose we have \( N \) people and we want to know how many ways are there to order them, i.e., to put them in a line from first to last.

Well, there are \( N \) ways to choose the first person in the line. There are \( N - 1 \) ways to choose the second person (the first person is already in line), so there are \( N \cdot (N - 1) \) ways to choose the first two people (multiplication principle again). Now there are \( N - 2 \) people
left, so there are \( N - 2 \) ways to choose the third person, \( N - 3 \) ways to choose the fourth, and so on. If we have chosen \( M \) people then the choice of the next person can be made in any of \( N - M \) ways. To put all \( N \) people in a line, we have

\[
N \cdot (N - 1) \cdot (N - 2) \ldots 3 \cdot 2 \cdot 1
\]

ways to choose. This function is called the “factorial” and is denoted

\[
F(N) = N \cdot (N - 1) \cdot (N - 2) \ldots 3 \cdot 2 \cdot 1 = N!.
\]

Factorial functions grow much more quickly than exponential functions.

There are (of course) functions that grow even faster than factorial.

### 1.6 Ordering by rate of growth

The list of functions above are distinguished by their growth rate in the “long-term”, that is, by how fast they get large as \( N \) gets large. We can even give an ordering from slow to fast as follows:

\[
\begin{align*}
\text{Zoo} \\
\text{Linear} & : L(N) = N \\
\text{Quadratic} & : Q(N) = N^2 \\
C(N) & = N^3 \\
& \vdots \\
P(N) & = 1.5^N \\
S(N) & = 2^N \\
& \vdots \\
F(N) & = N! \\
E(N) & = N^N
\end{align*}
\]

For each of these, if we plot one then we can not see much of the growth of the previous ones on the same scale. There are many more functions we can add that slip into this list e.g., \( N^{2.5} \) or \( 1.7^N \), ….

### 1.7 Disclaimer

The functions we have looked at all share the property of being increasing functions. As \( N \) gets larger, they get larger. Not every function does this. In fact, functions can have much more radical behavior, as we will see soon.
2 Exponential Growth, Revisited

Since exponential functions are so common, they deserve a second look. Looking at the graphs of exponential functions above, they all have an interesting property. They seem to “start slow”, then “explode”, i.e., the part of the function near zero is small and then at some point the graph shoots up and quickly leaves the top of the graph. This occurs because the rate of growth of an exponentially growing function, the amount it changes between \( N \) and \( N + 1 \) depends on how big it is at \( N \). The bigger the function gets, the faster it grows and a “feedback” loop quickly makes the function grow huge very suddenly.

This leads us to an important realization.

**EXPONENTIAL GROWTH CAN NOT GO ON FOREVER.**

This is a practical, rather than mathematical, statement. Even a linearly growing population will eventually fill up its environment. But because the rate of growth for exponentially growing functions continues to increase as the function increases, for exponential growth the crisis will come much more suddenly, and hence, be much more dangerous.

This is why, when we hear that Carbon dioxide levels are growing at two percent per year (i.e., like \( 1.02^N \), exponentially), we should get frightened. Will the world come to an end one morning? Maybe, maybe not, I don’t know. But I do know that this trend can NOT go on forever—it must change, either through some change in our behavior, some shift in the cycle that moves carbon dioxide out of the atmosphere, the collapse of industrial society and extinction of humans, or something else completely unexpected. Something has to give and the change will come on us quite suddenly.

Not that this is always bad. Before 2007, housing prices in the U.S. had been increasing exponentially for quite a while. A wise investor would realize this and “sell short”, that is, make investments that would benefit if housing prices decreased. This is what investor’s call a “bubble”, and bubbles always burst. So while a lot of people lost (and continue to lose) a lot of money in the crash of 2008, a few people made financial bets that the collapse would happen and they made a bundle.

3 One More Type of Function

A more unusual type function that is not in our list yet, but that you have probably seen, are the “logarithm” functions. For example, we let

\[ D(N) = \text{number of digits necessary to represent the integer } N. \]

Then for \( 0 \leq N \leq 9 \), \( D(N) = 1 \), for \( 10 \leq N \leq 99 \), \( D(N) = 2 \), for \( 100 \leq N \leq 999 \), \( D(N) = 3 \) and so on. We can see from this that the \( D(N) \) gets larger as \( N \) gets larger, but extremely slowly. For example, \( D(1,000,000,000) = 10 \), or \( D(\text{one trillion}) \) is only 10. The function \( D(N) \) is related to the logarithm function base 10 or \( \log_{10} \).
We can define \( \log_{10}(N) \) by the following:

\( \log_{10}(N) \) is the power you must raise 10 by to get \( N \)

So \( \log_{10}(100) = 2 \) because \( 10^2 = 100 \). Likewise \( \log_{10}(1000) = 3 \) because \( 10^3 = 1000 \). For numbers that are not a power of 10, the \( \log_{10} \) value will not be an integer. So \( \log_{10}(267) \) is between 2 and 3, because \( 100 < 267 < 1000 \). In fact, \( \log_{10}(267) \approx 2.426 \) because \( 10^{2.426} \approx 267 \).

When we begin to discuss ideas in computer science, we will see that it is much more natural to use \( \log_2 \) in place of \( \log_{10} \). The definition of \( \log_2(N) \) the power you must raise 2 by to get \( N \), so \( \log_2(16) = 4 \) because \( 2^4 = 16 \) and \( \log_2(97) \) is between 6 and 7 because \( 2^6 = 64 < 97 < 128 = 2^7 \). In fact, \( \log_2(97) \approx 6.6 \) because \( 2^{6.6} \approx 97 \).

### 3.1 A Property of Logs

Logarithm functions seem to pop up in surprising places (we will see them often when counting things in computer science).

In our definition of logarithm base 10, we noted that \( \log_{10}(1000) = \log_{10}(10^3) = 3 \) because you raise 10 to the third power to get 1000. In general,

\[
\log_{10}(10^N) = N.
\]

Noting that \( \log_{10}(10) = \log_{10}(10^1) = 1 \), we see that

\[
\log_{10}(10^N) = N \log_{10}(10).
\]

This rule generalizes as follows:

\[
\log_{10}(x^b) = b \log_{10}(x),
\]

that is, Log turns exponentiation into multiplication. This same rule holds for any base, so, for example,

\[
\log_2(5^3) = 3 \log_2(5).
\]

This rule leads to one of the great things about logarithms. Logs make “exponentially” large numbers small. We can use this to get a handle on functions that grow very fast.

We saw above that exponential functions grow very quickly. So for large \( N \), \( S(N) = 2^N \) is huge and its graph doesn’t give much useful information about the value of \( S(N) \) for small values of \( N \). However, we now know that

\[
\log_{10}(S(N)) = \log_{10}(2^N) = N \log_{10}(2) \approx N \cdot 0.301
\]

and this is a linear function with slope \( \log_{10}(2) \approx 0.301 \).
Looking at the graph of $\log_{10}(S(N))$ we can see what is going on for both small and large values of $N$. This idea is sometimes called graphing on a “logarithmic” scale and it can be very useful in identifying if exponential growth is occurring. For example, if we look at a graph of the number of internet host computers from 1980 to 2000, we see a curve that looks a bit like exponential growth (relatively small below 1995, then rapid growth until 2000). (Data from [https://www.isc.org/solutions/survey/history](https://www.isc.org/solutions/survey/history)). Plotting the $\log_{10}$ of the number of internet hosts from 1980 to 2000, we see a curve that is pretty close to a straight line from 1980 to 1986, then after a jump, it is again straight from 1990 to 2000. So we suspect that we can make a good approximation of the number of internet hosts using an exponential function. That is, we can honestly say that from 1990 to 2000, “the number of internet hosts grew exponentially”.
Can we use this graph to predict the future number of internet hosts (i.e., will the number of internet hosts keep growing exponentially)? Looking at the graph of the number of internet hosts from 2000 to 2009, we see that it does indeed look like the number of internet hosts maintains its growth. However, if we look at the $\log_{10}$ of the number of internet hosts, then we see that the curve after 2000 is not rising as rapidly as before 2000. From this graph it seems that the rate of growth is slowing down.
Exercises:

1. How many ways are there to choose a subcommittee of 4 different people from a group of \( N \) people (order of choice does not matter).

2. How many ways are there to pick a president, vice-president and treasurer from a class of \( N \) students?

3. How many ways are there for two people (person A and person B) to choose their Facebook friends from a group of \( N \) people? (Hint: Don’t try to list the possibilities. Instead, think of going through the list of \( N \) people and assigning them to be a friend of person A only, a friend of person B only, a friend of both or a friend of neither.)

4. Suppose a teacher has a class of \( N \) students.

   (a) The teacher wants to ask five questions during lecture and call on individual students for the answers. How many ways are there to choose a student for each of the questions? (Hint: There are at least two correct answers! What additional piece of information do you need and what is the answer in each case?)

   (b) The class has four discussion sections. How many ways are there to assign the students to discussion sections, (it doesn’t matter how many students are in each discussion section)?

5. Have a calculator compute \( \log_{10}(0.1) \). Explain why the answer is negative. (If you do not have a calculator that can compute log base 10, go to www.google.com and type (for example) “log(0.1)=” in the search box. Google will compute the \( \log_{10}(0.1) \) for you.)

6. The populations of North Carolina and Massachusetts from 1800 to 2000 given below in millions of people (U.S. Census data).

<table>
<thead>
<tr>
<th>Year</th>
<th>Pop.of Mass</th>
<th>Pop.of N.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1800</td>
<td>0.423</td>
<td>0.478</td>
</tr>
<tr>
<td>1820</td>
<td>0.523</td>
<td>0.638</td>
</tr>
<tr>
<td>1840</td>
<td>0.738</td>
<td>0.753</td>
</tr>
<tr>
<td>1860</td>
<td>1.231</td>
<td>0.993</td>
</tr>
<tr>
<td>1880</td>
<td>1.783</td>
<td>1.399</td>
</tr>
<tr>
<td>1900</td>
<td>2.805</td>
<td>1.893</td>
</tr>
<tr>
<td>1920</td>
<td>3.852</td>
<td>2.559</td>
</tr>
<tr>
<td>1940</td>
<td>4.317</td>
<td>3.571</td>
</tr>
<tr>
<td>1960</td>
<td>5.149</td>
<td>4.556</td>
</tr>
<tr>
<td>1980</td>
<td>5.737</td>
<td>5.88</td>
</tr>
<tr>
<td>2000</td>
<td>6.349</td>
<td>8.049</td>
</tr>
</tbody>
</table>
Make a “Logarithmic scale” graph of the populations. That is, compute the log base 10 of the population for each year, then plot these numbers with year on the horizontal axis and log of the population on the vertical axis. (If you do not have a calculator that can compute log base 10, go to www.google.com and type (for example) “log(0.423)=” in the search box. Google will compute the $\log_{10}(0.423)$ for you.) You do not have to turn in this graph, just answer the following question:

Is the population growing exponentially in Massachusetts or North Carolina? (justify your answer in a couple of sentences.)

7. In his recent “Wunderground” blog on Tropical weather, Jeff Masters described the effectiveness of the forecasts for Hurricane Bill which passed near New England in August of 2009. (See http://www.wunderground.com/blog/JeffMasters/archive.html for August, 2009)

All of the predictions depend on computer models. For the UKMET model the errors in prediction of the center of the storm are given below as a function of the hours in the future of the prediction.

<table>
<thead>
<tr>
<th>Hours in future</th>
<th>Nautical miles of error</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td>48</td>
<td>110</td>
</tr>
<tr>
<td>72</td>
<td>180</td>
</tr>
<tr>
<td>96</td>
<td>310</td>
</tr>
<tr>
<td>120</td>
<td>470</td>
</tr>
</tbody>
</table>

(a) Does the error grow at an exponential rate? Justify your answer.

(b) Write a sentence or two about what your answer to part a implies about the usefulness of these predictions?