In the last lecture, we found that if we have some way to map from a 3D plane to a sphere, then we can reduce to the Minnesota Lake Problem. Our goal is to lift and spread out lakes.

One way to do this is by a Mobius transform:

\[ f(x) = \frac{ax+b}{cx+d} \]

If you define the coefficients properly you can get conformal maps.

**Inverse Map:** \( \mathbb{R}^d \to \mathbb{R}^d \)

\[ R(v) = \frac{v}{v^Tv} = \frac{v}{||v||^2} \]

where \( v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \).

If we had divided by \( ||v|| \) we would have normalized \( v \) to be on the unit circle. However, because we are dividing by \( ||v||^2 \) we are inverting the space, such that infinity maps to the origin. Note that using the inverse map is much like stereographic projection, except there is no need to go to higher dimension.

A sphere can be defined as being all points equidistant to a single fixed point. Thus, the general formula for a sphere is

\[ ax^T x + bx^T v_0 + c = 0 \]

This is equivalent to the more generally recognizable formula:

\[ (x_1 - a_1)^2 + (x_2 - a_2)^2 + \ldots + (x_d - a_d)^2 = r^d \]

where \( r \) is the (positive) radius, and \( a \) is the center. Rearranging the first formula gives us

\[ x^T x + \frac{b}{c} x^T v_0 + \frac{c}{a} = 0 \]

Thus, the radius folds into \( \frac{c}{a} \).

Now, we have a collection of points on sphere \( x \) and we want to say that \( Rx : ax^T x + bx^T v_0 + c = 0 \) preserves spheres.
We know that under $R$ the following mapping exists

$$x \rightarrow \frac{x}{||x||^2}$$

Thus, the following are true:

$$x \rightarrow \frac{x}{x^T} \cdot x^T \rightarrow \frac{x^T}{x^T}$$

Substituting these into the equation,

$$a \frac{x^T}{x^T} \cdot x + b \frac{x^T}{x^T} \cdot v_0 + c = 0$$

Simplifying and then multiplying the entire equation by $x^T x$ gives

$$a + bx^Tv_0 + cx^Tx = 0$$

Now note that the equation has mapped to itself—the $a$ and $c$ coefficients have effectively switched places. Therefore, this mapping preserves spheres. We now have 3 maps that preserve spheres:

$$X \rightarrow x - a \cdot x \rightarrow ax \cdot R(x) \rightarrow \frac{x}{x^T}$$

Therefore, composition of these map functions will also preserve spheres.

Stereographic Projection:

$$\mathbb{R}^d \rightarrow S^d \subset \mathbb{R}^{d+1}$$
We are projecting onto the unit sphere in a higher dimension. Thus, the sphere is defined as

$$s^d = v : v^Tv = 1$$

The point at the top of the unit sphere is defined as

$$v_t = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

Thus, an example of a line from a point $x$ in 2D to the top of the sphere in 3D is defined as

$$\begin{pmatrix} x \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ -1 \end{pmatrix}$$

A parameterized version of this equation is

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} x \\ -1 \end{pmatrix} = \begin{pmatrix} 0 + tx \\ 1 - t \end{pmatrix}$$
To find the point of intersection of this line with the sphere, we need to find the value of $t$ such that the distance from the point on the line to the center of the sphere is 1 since it is a unit sphere. Thus,

$$\begin{pmatrix} tx \\ 1 - t \end{pmatrix} = \begin{pmatrix} t x \\ 1 - t \end{pmatrix} = 1$$

$$t^2 x^T x + (1 - t^2) = 1 t^2 x^T x + (1 - 2t + t^2) = 1 t^2 x^T x - 2t + t^2 = 0 tx^T x - 2 + t = 0 t = \frac{2}{1 + x^T x}$$

Substituting $t$ back into $\begin{pmatrix} 0 + tx \\ 1 - t \end{pmatrix}$ yields

$$\Pi(x) = \begin{pmatrix} 2x \\ \frac{1 - x^T x}{1 + x^T x} \\ \frac{x^T x - 1}{1 + x^T x} \end{pmatrix}$$

If we think of space as an infinite sphere, the infinite sphere maps to a finite sphere through stereographic projection, where the equator is the fix point.

Now, we want to write $\Pi(x)$ as an inverse mapping. We begin by defining

$$G(u) = \frac{2(u - e)}{(u - e)^T (u - e)} + e$$

where

$$e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

is the north pole of the sphere. Note that $G(u)$ is a shift of $u$ to the center of the sphere, followed by an inverse mapping, then another shift by $e$. Also note that $G(u) = 2R(u - e) + e$.

Let $L : x \rightarrow \begin{pmatrix} x \\ 0 \end{pmatrix}$. Prove: $\Pi(x) = G(L(x))$. 
\[ \Pi(x) = G(L(x)) \]
\[ = \frac{2(L(x) - e)}{(L(x) - e)T(L(x) - e)} + e \]
\[ = \frac{2(L(x) - e)}{(x^T - 1)(x - 1)} + e \]
\[ = \frac{2(L(x) - e)}{1 + x^T x} + e \]
\[ = \begin{pmatrix} \frac{2x}{1 + x^T x} \\ \frac{x^T x - 1}{1 + x^T x} \end{pmatrix} \]

Thus, we can write \( \Pi(x) \) as a shift, inverse mapping, and another shift, and it maps a 3D sphere to a 3D sphere.

**Involution:** \( G(G(u)) = u \)

**Proof:**

We know from before that the equation for \( G \) is: \( G(u) = 2 * R(u - e) + e \). Thus,

\[
\begin{align*}
G(u) - e &= 2 * R(u - e) \\
R(G(u) - e) &= R(2 * R(u - e)) \\
R(G(u) - e) &= \frac{1}{2} * R(R(u - e)) \\
R(G(u) - e) &= \frac{1}{2} *(u - e) \\
2 * R(G(u) - e) &= u - e \\
2 * R(G(u) - e) + e &= u \\
G(G(u)) &= u
\end{align*}
\]

This family of maps is more general than stereographic projection; we can shift it around, scale the size of the mapping ball rather than using the unit sphere, etc., in order to spread out "lakes" across the sphere.
Given a $d$-dimensional sphere $S^d$ with caps on the sphere $\{B_1, B_2, \ldots, B_n\} \subset S^d$ and a random great circle $C$, we expect $\mathbb{E}(|B_i \cap C|) = O(n^{1-\frac{1}{d+2}})$.

For the geometric separator, we want to prove the following theorem:

Let $\{B_1, B_2, \ldots, B_n\}$ be a collection of non-overlapping balls in $\mathbb{R}^d$. $\exists$ a sphere $S$ such that:

1) $|\{B_i | B_i \subset \text{interior}(S)\}| \leq \frac{d+1}{d+2}n$

2) $|\{B_i | B_i \subset \text{exterior}(S)\}| \leq \frac{d+1}{d+2}n$

3) $|\{B_i \cap S \neq \emptyset\}| \leq \frac{d+1}{d+2}n$

We want a conformal map: $\phi: \mathbb{R}^d \rightarrow S^d$ such that:

1) $B_i \rightarrow \phi(B_i)$ is a sphere on $S^d$ (where $B_i$ has center $\hat{B}_i$)

2) The center of $S^d$ satisfies:

$\forall$ hyperplanes $H$ passing through the origin $O$, the hyperplane has 2 sides, $H^+$ and $H^-$ (say the top and bottom hemispheres) such that:

$|H^+ \cap \{\phi(\hat{B}_1), \ldots, \phi(\hat{B}_n)\}| \leq \frac{d+1}{d+2}n$

$|H^- \cap \{\phi(\hat{B}_1), \ldots, \phi(\hat{B}_n)\}| \leq \frac{d+1}{d+2}n$

Center point

In 1D, the center point is simply the 1/2 median.

A $\delta$-median is defined as: $\forall \delta \leq 1/2$, and a set of values $\{a_1, \ldots, a_n\}$, $b$ is a $\delta$-median if:

$|\{a_i \leq b\}| > \delta n$

$|\{a_i \geq b\}| > \delta n$

$(1-\delta)n \geq |\{a_i \leq b\}|$
In 2D, the median is a line that splits the values into two.

We want to find a point in the plane such that any line through that point gives a good median; this is defined as the center point.

**Theorem:** \( \forall \) point set \( p = \{p_1, ..., p_n\} \in \mathbb{R}^d \exists \) a point \( c \in \mathbb{R}^d \) such that \( c \) is a \( \frac{1}{d+1} \)-centerpoint.

Coming back to our problem, we have centers of balls \( \{\hat{B}_1, ..., \hat{B}_n\} \in \mathbb{R}^d \).
\( \{p_1, ..., p_n\} \in \mathbb{R}^d \) and we want to map these points to the sphere via \( \phi(\{p_1, ..., p_n\}) \rightarrow S^d \) so that the center of \( S^d \) is a \( \frac{1}{d+2} \)-centerpoint of \( \{\phi(p_1), \phi(p_2), ..., \phi(p_n)\} \).

When a plane passing through the centerpoint \( c \) is intersected with the unit sphere, it will be a circle. What we want to do now is relocate \( c \) this circle such that it lines up with the equator of the sphere, such that \( \frac{1}{d+2} \) of the caps will be on one side of the plane and the rest will be on the other. Thus, when we do the inverse mapping under this series of transformations, it will result in a cut that has some points in the interior of \( S \) and some outside.
So we can rotate to get $c$ below the plane:

Note that the inverse of the inverse mapping of $c$ is a smaller circle on the plane of $\Re^d$. Thus, if we then dilate this circle to be in line with the unit circle, the inverse mapping of $c$ will move up to become the equator:

Thus,

$$r = \sqrt{\frac{1-||c||^2}{1+||c||^2}}$$

We need to dilate

$$\frac{1}{r} = \sqrt{\frac{(1+||c||)^2}{1-||c||^2}}.$$
\{\phi(p_1), \phi(p_2), ..., \phi(p_n)\} \rightarrow c \rightarrow \theta(c) \rightarrow \Pi^{-1} \rightarrow \text{dilate by } \frac{1}{r} \rightarrow \Pi

We can get a centerpoint \(c\) from the mapped points, rotate \(c\) by \(\theta\) to be below the plane, which is the stereographic projection only below the plane, dilate by \(\frac{1}{r}\) to arrive at the inverse projection in 2D.

Note that a hyperplane through \(c\) cuts a diameter of the lower circle. After dilation, the diameter is still a diameter; thus, the image passes through the center.