Basic Intuitions of Geometry

- Points in space
- Shapes
- Distance (the most fundamental) - \( \text{dist}(p, q) \) is the Euclidean distance between two points \( p \) and \( q \):
  \[
  \text{dist}(p, q) := \sqrt{(p_x - q_x)^2 + (p_y - q_y)^2}
  \]

A dimension is a parameter that precisely measures the conceptual and visual complexity of any geometric object. As the dimension grows higher, the structure of the object gets richer. For example, consider a point in space, its dimensionality is 0. If that point were dragged left or right, it would create a line in one-dimensional space, \( \mathbb{R}^1 \). Now if this line is dragged up or down, it now creates a square in two-dimensional space. If dragged backward or forward, the square becomes a cube with dimensionality 3, hence, a 3D object.

1-Dimensional Range Searching

Input: a set of points, \( P = \{p_1, p_2, \ldots, p_n\} \in \mathbb{R}^1 \)

Question 1: \( \forall i, \) Compute \( p_i \)'s nearest neighbor in \( P \).

Question 2: Given \( k \in \mathbb{N}, \forall i, \) Compute \( p_i \)'s \( k \) closest neighbors in \( P \).

Question 3: Construct a query structure, then, given a query point \( q \) in \( \mathbb{R}^1 \), compute \( q \)'s nearest neighbor.

Questions 1 and 2 are typically referred to as offline query problems. An offline query is aware of all data prior to making decisions on it. Question 3 is an
example of an *online* query problem; it has to process each data entry in turn without detailed knowledge of future entries.

The sorting of points can be done in one dimension. For Question 1, the best sorting complexity is $O(n \log n)$ because all the points can be sorted collinearly and then only the neighbors to the left and to the right of $p_i$ have to be checked. The same can be said for Question 2, however, the complexity becomes $O(n \log n + kn)$ since the output depends on $k$, the number of nearest neighbors. This is often called *output sensitive complexity*, meaning that the complexity is sensitive to the size of the output.

In Question 3, the complexity consists of three parts:

1. the time to construct the query structure,
2. the space needed,
3. the query time.

The structure can easily be constructed using quicksort in $O(n \log n)$ time. The quicksort algorithm can be viewed as a *random incremental* algorithm that produces a binary search tree needing $O(n)$ space. So, the complexity for Question 3 is $O(n \log n)$. Note that a random incremental algorithm is an algorithm that randomly adds data entries one at a time updating the solution after each addition.

**Voronoi Diagrams**

*Definition* Let $P := \{p_1, p_2, \ldots, p_n\}$ be a set of $n$ distinct points in the plane. The Voronoi diagram of $P$, denoted $Vor(P)$, is the subdivision of the plane into $n$ Voronoi regions, denoted $V(p_i)$. There is one region for each point $p_j \in P$, with the property that a point $q$ lies in the region corresponding to $p_i$ if and only if $\text{dist}(q, p_i) < \text{dist}(q, p_j)$ for each $p_j \in P$ with $j \neq i$. In other words, a point $q$ laying in $V(p_i)$ is closest to that $p_i$ than any other $p$ in the diagram. Remember from the first page, $\text{dist}(q, p)$ denotes the Euclidean distance between two points $p$ and $q$. 


For one point, the Voronoi region is all the space.

\[ V(p_1) \]

For two points, the line segment connecting them is split in half by a perpendicular bisector creating two half-planes.

\[ p_1 \]
\[ p_2 \]
\[ V(p_1) \]
\[ V(p_2) \]

**Theorem** Let \( P \) be a set of \( n \) points in the plane. If all the points are collinear then \( \text{Vor}(P) \) consists of \( n-1 \) parallel lines. Otherwise, \( \text{Vor}(P) \) is connected and its edges are either segments or half-planes.

Notice how this theorem applies to the previous two-point example and the following example when all the points in \( P \) are collinear. Note the number of parallel lines.

\[ \text{Vor}(P) \]
\[ p_1 \]
\[ p_2 \]
\[ p_3 \]
\[ p_4 \]
\[ p_5 \]
\[ V(p_1) \]
\[ V(p_2) \]
\[ V(p_3) \]
\[ V(p_4) \]
\[ V(p_5) \]

Now, consider three non-collinear points in \( P \) (points that do not lie on a straight line). It can easily be seen that \( \text{Vor}(P) \) is connected at one central vertex, created where the three perpendicular bisectors intersect, and has three edges.

\[ V(p_1) \]
\[ V(p_2) \]
\[ p_1 \]
\[ p_2 \]
\[ p_3 \]
\[ V(p_3) \]
Another important property of the Voronoi diagram follows from the *empty circle* property. This property says that for a point \(q\), the largest empty circle of \(q\) with respect to \(P\), denoted \(C_P(q)\), is the largest circle with \(q\) as its center that does not contain any point of \(P\) in its interior. This is also known as a *\(P\)-free* circle.

\[
C_P(q)
\]

**Theorem**  
For the Voronoi diagram \(\text{Vor}(P)\) of a set of points \(P\) the following holds:

- A point \(q\) is a vertex of \(\text{Vor}(P)\) if and only if its largest empty circle \(C_P(q)\) contains three or more \(p\) points on its boundary.

- The bisector between points \(p_i\) and \(p_j\) defines an edge of \(\text{Vor}(P)\) if and only if there is a point \(q\) on the bisector such that \(C_P(q)\) contains both \(p_i\) and \(p_j\) on its boundary but no other point.

The following are a few examples that employ the vertex and edge properties of the previous theorem.

1. \(V(p_i) = \{ q : \|q - p_i\| < \|q - p_j\|, \forall j \neq i \}\)
2. \( V(p_i, p_j) = \{ q : \| q - p_i \| = \| q - p_j \| < \| q - p_k \|, \forall k \neq i, k \neq j \} \)

![Diagram showing the Voronoi region for points \( p_i \) and \( p_j \).]

3. \( V(p_i, p_j, p_k) = \{ q : \| q - p_i \| = \| q - p_j \| = \| q - p_k \|, \forall s \neq i, j, k \} \)

![Diagram showing the Voronoi region for points \( p_i \), \( p_j \), and \( p_k \).]

Question: When do \( V(p_i) \) and \( V(p_j) \) share a one-dimensional boundary?

When \( V(p_i, p_j) \) is not empty. This suggests that there is at least one edge in \( V(p_i, p_j) \). To guarantee that the two points are in neighboring regions, some ball passing through them must be empty. So, the two regions are neighbors if and only if there exists a ball passing through them is empty of any other points in \( P \).

Note that in one-dimension there are only two types of regions, one-dimensional regions and zero-dimensional regions.

The Post Office Problem is a famous example of using Voronoi diagrams.

- Post offices are represented by a set of points, \( P = \{ p_1, p_2, ..., p_n \} \) \( \in \mathbb{R}^d \).
- The Voronoi regions represent zip code areas for each post office.
- Each household is a query point \( q \) in a zip code.
Given a set \( P \) of post offices, determine...

1. a household \( q \) that is closest to only one post office.
2. a household \( q \) that is closest to two post offices (on the boundary of 2 regions).
3. a household \( q \) that is equidistant to three post offices.

These are only a few questions that could be answered simply by knowing the properties of Voronoi diagrams and the distance formula.

**Delaunay Triangulations**

*Definition* A triangulation on a set of \( P \) points is a planar subdivision whose bounded faces are triangles and whose vertices are the points of \( P \). Any triangulation obtained by adding edges to the Delaunay graph of \( P \) by connecting two points whose regions share a Voronoi boundary is called a Delaunay triangulation.

Let \( P = \{p_1, p_2, \ldots, p_n\} \) be a set of points in a plane, the Delaunay graph of \( P \), denoted \( DG(P) \), is the dual graph of the Voronoi diagram of \( P \). To construct \( DG(P) \), first construct the Voronoi diagram of \( P \). Then, for each \( p_i \) and \( p_j \) whose regions share a Voronoi boundary, create an arc connecting them. Finally, straighten the arcs into line segments.
Note that if the points in \( P \) are randomly distributed there is a very small chance that four points will lay on the same circle. When no four points are on a circle, \( P \) is said to be in general position. If \( P \) is in general position, all vertices of the Voronoi diagram have degree three. Thus, all bounded faces of the Delaunay graph are triangles. This is why the Delaunay graph of \( P \) is often called the Delaunay triangulation of \( P \). Interestingly, the number of triangles and the number of edges is the same in any triangulation of \( P \).

**Theorem**  Let \( P \) be a set of \( n \) points in the plane, not all collinear, and let \( k \) denote the number of points in \( P \) that lie on the boundary of the convex hull of \( P \). Then any triangulation of \( P \) has 2\(n\)-2-\(k\) triangles and 3\(n\)-3-\(k\) edges.

Properties concerning vertices and edges of a Delaunay graph are very similar to the properties of vertices and edges in Voronoi diagrams.

**Theorem**  Let \( P \) be a set of points in the plane.

- Three points \( p_i, p_j, p_k \in P \) are vertices of the same face of the Delaunay graph of \( P \) if and only if the circle through \( p_i, p_j, p_k \) contains no point of \( P \) in its interior.
- Two points \( p_i, p_j \in P \) form an edge of the Delaunay graph of \( P \) if and only if there is a closed disc \( C \) that contains \( p_i \) and \( p_j \) on its boundary and does not contain any other point of \( P \).

Furthermore, if there exists a triangle, \( \triangle p_i p_j p_k \), then the Voronoi region, \( V(p_i p_j p_k) \), is not empty.
As it is obvious to see, the Voronoi diagram and the Delaunay triangulation of a set of points $P$ are very closely related. This is because (in $\mathbb{R}^2$) they are dual graphs of each other. Consequently, since a Voronoi diagram is a planar graph, the Delaunay triangulation is also a planar graph.

**Theorem**  The Delaunay graph of a planar point set is a plane graph.

The Delaunay graph of $P$ has a face for every vertex of $\text{Vor}(P)$ and the edges around a face correspond to the Voronoi edges incident to the corresponding Voronoi vertex.

Recall that a graph $G = (V, E)$ is planar if there exists $\pi$, where $V \in \mathbb{R}^2$. Hence, the embedding of $(i, j) \in E$ to a segment $[\pi(i), \pi(j)]$ such that no two edge segments cross in a plane other than at endpoints.

The number of edges in a Voronoi diagram or Delaunay triangulation is bounded by $O(n)$. This can easily be shown using Euler’s formula for planar graphs.

$$|V| - |E| + |F| = 2$$
Also, in a planar graph, every face has at least three edges and every edge “sees” at least two faces. So, \( \frac{|F|}{|E|} \leq \frac{2}{3} \) implies \( \frac{2}{3}|E| \geq |F| \). Applying this to Euler’s formula, we get \( |V| - |E| + \frac{2}{3}|E| \geq 2 \).

**Example:**

\[
\begin{align*}
V &= 4 \\
E &= 6 \\
4 - 6 + \frac{2}{3}(6) &\geq 2 \\
2 &\geq 2
\end{align*}
\]

**Theorem** For \( n \geq 3 \), the number of vertices in the Voronoi diagram of a set of \( n \) points in the plane is at most \( 2n-5 \) and the number of edges is at most \( 3n-6 \).