CS480 Solutions to Homework Assignment 1

1. We want to determine the transformation matrix for a reflection around an arbitrary line \( y = mx + b \). One way to solve this is to translate and rotate the problem into an easier coordinate system. If we translate and rotate the line to align it with the \( x \) axis, then we can employ the standard transform for reflection about the \( x \)-axis.

The homogeneous coordinate representation is more convenient here because we need to include translation in our formulation. Thus we can build a general reflection about a line as a concatenation of linear transforms. We begin by translating to the line’s \( y \) intercept to the origin using a translation matrix:

\[
T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}.
\] (1)

We then need to rotate the line to align it with the \( x \) axis:

\[
R = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\] (2)

The sine and cosine for alignment can be found directly from the line equation:

\[
\sin \theta = \frac{-m}{\sqrt{1 + m^2}} \quad \text{and} \quad \cos \theta = \frac{1}{\sqrt{1 + m^2}}.
\] (3)

Now that we have a way to get between the line’s coordinate system and the world coordinate system, we can employ reflection about the \( x \)-axis:

\[
\text{Reflect}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\] (4)

We then inverse rotate, back into the line’s orientation:

\[
R^{-1} = R^T = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\] (5)

And finally, translate from the origin back to the line’s \( y \) intercept:

\[
T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}.
\] (6)

Thus we build our arbitrary reflection as a concatenation of transforms. We first align the coordinate system of the line with the \( x \) axis, we then reflect, and then we return to the original coordinate system:

\[
\text{Reflect}_{\text{line}} = T_2 R^T \text{Reflect}_x R T_1
\] (7)
2. We need to show that $\mathbf{SR} = \mathbf{RS}$, only if $s_x = s_y$, or $\theta = n\pi$ for integer values of $n$. Start by multiplying the matrices:

\[
\mathbf{SR} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} \cos \theta & - \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} s_x \cos \theta & -s_x \sin \theta \\ s_y \sin \theta & s_y \cos \theta \end{bmatrix},
\]

and

\[
\mathbf{RS} = \begin{bmatrix} \cos \theta & - \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} = \begin{bmatrix} s_x \cos \theta & -s_y \sin \theta \\ s_x \sin \theta & s_y \cos \theta \end{bmatrix}.
\]

If $\mathbf{SR} = \mathbf{RS}$, then clearly

\[
\begin{bmatrix} s_x \cos \theta & -s_x \sin \theta \\ s_y \sin \theta & s_y \cos \theta \end{bmatrix} = \begin{bmatrix} s_x \cos \theta & -s_y \sin \theta \\ s_x \sin \theta & s_y \cos \theta \end{bmatrix}.
\]

Note that the diagonal elements of both matrices are the same, while the off-diagonal elements are different. This means that for the matrices to be equal, it must be the case that

\[
-s_x \sin \theta = -s_y \sin \theta \quad \text{and} \quad s_y \sin \theta = s_x \sin \theta.
\]

This will be true when $s_x = s_y$, or when $\sin \theta = 0$ (which happens only when $\theta = n\pi$ for integer values of $n$). Otherwise, the off diagonal elements of the matrices will not be equal, and therefore $\mathbf{SR} \neq \mathbf{RS}$.

3. First, we will recall from the textbook that:

1. $\mathbf{R}(\theta)\mathbf{R}(\phi) = \mathbf{R}(\theta + \phi)$. Since this is true, any sequence of rotations can be rewritten as one rotation matrix.

2. $\mathbf{T}(a_x, a_y)\mathbf{T}(b_x, b_y) = \mathbf{T}(a_x + b_x, a_y + b_y)$. Since this is true, any sequence of translations can be rewritten as one translation.

In addition, we must prove that any rotation followed by a translation can be rewritten in terms of a different translation followed by the rotation; e.g., $\mathbf{R}(\theta)\mathbf{T}(a_x, a_y) = \mathbf{T}(b_x, b_y)\mathbf{R}(\theta)$. The proof of this is given as follows.

\[
\mathbf{R}(\theta)\mathbf{T}(a_x, a_y) = \begin{bmatrix} \cos \theta & - \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} \cos \theta & - \sin \theta & (a_x \cos \theta - a_y \sin \theta) \\ \sin \theta & \cos \theta & (a_x \sin \theta + a_y \cos \theta) \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 0 & (a_x \cos \theta - a_y \sin \theta) \\ 0 & 1 & (a_x \sin \theta + a_y \cos \theta) \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
= \mathbf{T}(b_x, b_y)\mathbf{R}(\theta)
\]

where, $b_x = a_x \cos \theta - a_y \sin \theta$ and $b_y = a_x \sin \theta + a_y \cos \theta$. 


4. (a) 

\[
M = \begin{bmatrix}
-1 & 0 & a \\
0 & -1 & b \\
0 & 0 & 1
\end{bmatrix}.
\]

4. (b) Rotate by 180° followed by translation by \((a, b)\).

4. (c) 

\[
P' = \begin{bmatrix}
1 & 0 & c_x \\
0 & 1 & c_y \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x_x & g_x & 0 \\
x_y & g_y & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
x_x & x_y & 0 \\
g_x & g_y & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & -c_x \\
0 & 1 & -c_y \\
0 & 0 & 1
\end{bmatrix} P.
\]

5. (extra credit) In this problem, we need to find a concatenation of basic linear transforms (rotations, scalings, or translations) that is equivalent to the x-direction shearing matrix. The solution to this problem requires two rotations and one scale:

\[
Shear_x = R(\gamma)SR(\theta) = R(\gamma) \begin{bmatrix}
s_x & 0 & 0 \\
0 & s_y & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Multiplying out \(SR(\theta)\) we get

\[
Shear_x = R(\gamma) \begin{bmatrix}
s_x \cos \theta & -s_x \sin \theta \\
s_y \sin \theta & s_y \cos \theta \\
\end{bmatrix}.
\]

We want to find \(R(\gamma)\) that rotates \(SR(\theta)\) into the \(x\) shear matrix:

\[
\begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}.
\]

In general, multiplying by a rotation matrix takes the form:

\[
\begin{bmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{bmatrix} \begin{bmatrix}
s_x \cos \theta & -s_x \sin \theta \\
s_y \sin \theta & s_y \cos \theta
\end{bmatrix} = \begin{bmatrix}
1 & a \\
0 & 1
\end{bmatrix}.
\]

The lower left corner of the resulting shear matrix must satisfy \(\beta s_x \cos \theta + \alpha s_y \sin \theta = 0\). This means that \(\alpha = s_x \cos \theta\) and \(\beta = -s_y \sin \theta\). Therefore the angle for the second rotation matrix is \(\gamma = \arctan \frac{-s_y \sin \theta}{s_x \cos \theta}\), and we need to scale the whole system by \(\rho = s^2_y \sin^2 \theta + s^2_x \cos^2 \theta\). We can now rewrite in terms of this rotation and scaling:

\[
\begin{bmatrix}
\alpha & -\beta \\
\beta & \alpha
\end{bmatrix} = \rho R(\gamma) = \begin{bmatrix}
s_x \cos \theta & s_y \sin \theta \\
-s_y \sin \theta & s_x \cos \theta
\end{bmatrix}.
\]
Plugging into Equation 21 and multiplying out, we get:

\[
\text{Shear}_x = \begin{bmatrix}
  s_x \cos \theta & s_y \sin \theta \\
  -s_y \sin \theta & s_x \cos \theta
\end{bmatrix} \begin{bmatrix}
  s_x \cos \theta & -s_x \sin \theta \\
  s_y \sin \theta & s_y \cos \theta
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  s_x^2 \cos^2 \theta + s_y^2 \sin^2 \theta & (s_y^2 - s_x^2) \cos \theta \sin \theta \\
  0 & s_x s_y (\cos^2 \theta + \sin^2 \theta)
\end{bmatrix}
\]

\[
= \begin{bmatrix}
  1 & a \\
  0 & 1
\end{bmatrix}.
\]

This gives us the equations:

\[
s_x^2 \cos^2 \theta + s_y^2 \sin^2 \theta = 1
\]

\[
(s_y^2 - s_x^2) \cos \theta \sin \theta = a
\]

\[
s_x s_y (\cos^2 \theta + \sin^2 \theta) = s_x s_y = 1
\]

From Equation 28, we see that \( s_x = \frac{1}{s_y} \). By plugging this into Equation 26, we find that

\[
\frac{1}{s_y^2} \cos^2 \theta + s_y^2 \sin^2 \theta = 1.
\]

This holds true when \( s_y = 1 \) or \( s_y = \frac{\cos \theta}{\sin \theta} = \cot \theta \). We prefer the case when \( s_y \neq 1 \), and plug into Equation 27:

\[
\left( \frac{\cos^2 \theta - \sin^2 \theta}{\sin^2 \theta - \cos^2 \theta} \right) \cos \theta \sin \theta = \frac{\cos^4 - \sin^4}{\cos \theta \sin \theta} = a.
\]

Using trigonometric identities, we find that

\[
\frac{\cos^4 - \sin^4}{\cos \theta \sin \theta} = \frac{2 \cos 2\theta}{\sin 2\theta} = 2 \cot(2\theta) = a.
\]

This means that

\[
\theta = \frac{1}{2} \arctan \frac{2}{a}.
\]

In summary, the \( x \) direction shear matrix can be written:

\[
\text{Shear}_x = \rho \mathbf{R}(\gamma) \mathbf{S}\mathbf{R}(\theta)
\]

where

\[
\theta = \frac{1}{2} \arctan \frac{2}{a}
\]

\[
\gamma = \arctan \left( -\frac{s_y \sin \theta}{s_x \cos \theta} \right)
\]

\[
s_x = \tan \theta
\]

\[
s_y = \cot \theta
\]

\[
\rho = s_y^2 \sin^2 \theta + s_x^2 \cos^2 \theta.
\]