5 Geometrical Basis of Sound

Geometry is frozen music.
—Goethe

5.1 Circular Motion and Simple Harmonic Motion

Suppose a pendulum swings back and forth above a turntable. The turntable has a marker, such as a small cone, placed on its surface (figure 5.1). The cone moves with uniform circular motion because a motor drives it in a circle at a constant speed. Now adjust the length of the pendulum so that it makes one full swing in the same time that the turntable makes one complete revolution, and release the pendulum at exactly the same moment the cone moves under it so that the two movements are synchronized. With the two motions so aligned, if we look directly edge-on at the turntable, the pendulum and the cone seem to have exactly the same left/right motion even though we know that the pendulum moves in a line while the turntable moves circularly. Intuitively, it looks like circular motion and simple harmonic motion are in some way equivalent if seen from the right vantage point. This train of thought suggests that we can use the geometry of circles to study simple harmonic motion and wave behavior.

5.2 Rotational Motion

Circular motion and simple harmonic motion are closely related. In fact, to understand circular motion is to understand sine waves, which are the basis of all musical sound. This section reviews information provided by geometry and trigonometry about circular motion.

5.2.1 Angular Displacement

The center of a rigid rotating body, such as a turntable, defines its axis of rotation as a point around which circular motion revolves. The angle through which the rigid body rotates about its axis of rotation is its angular displacement. Suppose a turntable rotates from an initial angle \( \theta_i \) to a final angle of \( \theta_f \). We say the turntable sweeps out the angle \( \theta \), defined as

\[
\theta = \theta_f - \theta_i.
\]

Angular Displacement (5.1)
Rotatable objects can turn either clockwise or counterclockwise.

*Counterclockwise angular displacement is taken to be positive, and clockwise angular displacement is taken to be negative.*

Thus, $\theta$ indicates counterclockwise rotation, and $-\theta$ indicates clockwise rotation.

### 5.2.2 Radians

It is common to use degrees to measure angular displacement or to refer to entire revolutions. One *revolution* returns a turntable to its initial position and equals 360°.

Suppose a turntable sweeps out an angle $\theta$ as shown in figure 5.2. As it does so, point Q traces out an arc of length $s$. Clearly, the length of $s$ grows if either its radius $r$ or the angle $\theta$ grows. In fact, we can show with elementary geometry that

$$\theta = \frac{\text{Arc length}}{\text{Radius}} = \frac{s}{r}. \quad (5.2)$$

When $s/r = 1$, that is, when the arc length is the same as the radius, the angle $\theta$ is equal to 1 *radian* (rad). Since both $s$ and $r$ are measures of distance, their ratio is a dimensionless number (because
the units in the numerator and denominator cancel out). A dimensionless number is a "pure number" unencumbered with physical significance.

If the point Q sweeps out one entire revolution of radius \( r \), its angular displacement will be \( \theta = 2\pi \) and its arc length \( s \) will equal the circumference of the circle, \( 2\pi r \).

Since one revolution equals \( 360^\circ \), we can equate degrees and radians. If \( \theta = 360^\circ \), then \( s = 2\pi r \) and

\[
\frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ rad},
\]

and \( 2\pi \text{ rad} = 360^\circ \). Solving for \( \text{ rad} \), we see that one radian is

\[
\text{ rad} = \frac{360^\circ}{2\pi} \approx 57.3^\circ.
\]

Radian (5.4)

This constant, the radian measure, allows us to use simple integers and ratios of integers to specify useful divisions of a circle.\(^1\) For example, the circumference of the circle is \( 2\pi \) radians, and a half circle (\( 180^\circ \)) is one half of that, exactly \( \pi \) radians. Similarly, one quarter of the circumference is \( \pi/2 \) radians, which is therefore \( 90^\circ \), the size of a right angle.

When angles are stated in radians, the constant \( \pi \) tends to drop out from equations, greatly simplifying calculations. Radian measure also simplifies calculation of the length of an arc. Solving (5.2) for \( s \) yields

\[
s = r\theta,
\]

Length of an Arc (5.5)

so we can get the length of \( s \) simply by multiplying the radius of its circle by the arc’s angle in radians.\(^2\)

5.2.3 Angular Velocity

Suppose a turntable starts at angle \( \theta_0 \) and rotates to angle \( \theta_r \) (figure 5.3). Then its angular displacement is \( \theta = \theta_r - \theta_0 \). Further, suppose the turntable performs this rotation in \( t \) seconds. Then its angular velocity is the angular displacement \( \theta \) divided by elapsed time \( t \):

\[
\omega = \frac{\theta}{t},
\]

Angular Velocity (5.6)

which we measure in SI units of rad/s.
Angular velocity is the rate at which angular displacement changes.

Compare (5.6) to linear velocity, which is the rate at which linear displacement changes. Counterclockwise angular velocities are positive, whereas clockwise angular velocities are negative. In (5.6) the symbol \( \equiv \) means "defined as." I use it to signify that I am defining \( \omega \) to have a particular meaning, namely, \( \theta/t \). Later, when I use \( \omega \), it will carry this significance.

Here's another way to calculate angular displacement. Suppose the turntable shown in figure 5.7 is set so that the cone is at its rightmost position, aligned with the x-axis. Then we start the turntable and start a timer at time \( t = 0 \). The turntable rotates counterclockwise at a constant rate of \( \omega \) rad/s, moving through angle \( \theta \) in time \( t \). Since the turntable rotates at a uniform speed, the size of the angle \( \theta \) grows at a constant rate. Therefore, the angular displacement \( \theta \) at time \( t \) is the angular velocity times the elapsed time \( t \):

\[
\theta = \omega t. \quad \text{(Angular Displacement with elapsed time) (5.7)}
\]

5.2.4 Angular Acceleration

If the turntable shown in figure 5.7 starts rotating with angular velocity \( \omega_0 \) and ends at time \( t \) with angular velocity \( \omega_f \), the change in angular velocity is \( \omega = \omega_f - \omega_0 \). If the change is not zero, the turntable exhibits angular acceleration \( \alpha \), which is change in angular velocity \( \omega \) through time \( t \):

\[
\alpha = \frac{\omega_f}{t} - \frac{\omega_0}{t} \quad \text{(Angular Acceleration) (5.8)}
\]

measured in SI units of (rad/s)/s = rad/s².

Angular acceleration is the rate at which angular velocity changes.

5.2.5 Rotational Speed

If a bicycle's wheel is turning once per second at a constant rate, and the tire's radius \( r = 0.5 \) m, how fast is the bicycle going? If the circumference of the wheel is \( c = 2\pi r = 3.14 \) m, then the velocity of the bicycle must be about \( 3.14 \) m/s. Every point on the circumference of the tire is also traveling at \( 3.14 \) m/s. Thus, for some radius \( r \) and some period of time \( T \), the rotational speed of a point on a circle is

\[
\nu = \frac{2\pi r}{T}. \quad \text{(Rotational Speed) (5.9)}
\]

5.2.6 Centripetal Acceleration

Speed doesn't imply direction, but velocity does. As a point on the circle travels, its direction changes moment by moment. So, even though the speed of a point on the circle remains uniform, its velocity changes from instant to instant because its direction changes.

Figure 5.4a shows a circle rotating through points \( p_1 \) and \( p_2 \). The velocity at these points can be drawn as vectors, \( v_1 \) and \( v_2 \), representing the linear velocity of each point. The difference of the two vectors is the change in velocity \( \Delta v = v_2 - v_1 \). The difference of two vectors can be shown by putting
their bases together and measuring the distance between their tips (Figure 5.4b). Similarly, the vector distance between \( p_1 \) and \( p_2 \) is \( \Delta r = r_2 - r_1 \) (Figure 5.4c). Since the length of \( v_1 = v_2 \) and the length of \( r_1 = r_2 \), triangle \( r_1r_2\Delta r \) and triangle \( v_1v_2\Delta v \) are both isosceles triangles.

Let's simplify things a bit. Since \( v_1 = v_2 \), let's define \( v = v_1 = v_2 \), and since \( r_1 = r_2 \), let's define \( r = r_1 = r_2 \) (Figures 5.4d and 5.4e). Note that the isosceles triangles in 5.4d and 5.4e have the same angle \( \theta \). So they are similar. From geometry we know that for similar triangles,

\[
\frac{\Delta v}{v} = \frac{\Delta r}{r}, \tag{5.10}
\]

For the next step, we can make a simplifying assumption. First, let \( \Delta t = t_2 - t_1 \), the time it takes for \( p_1 \) to get to \( p_2 \). Now, for small angles \( \theta \),

\[
\Delta r \approx v \cdot \Delta t. \tag{5.11}
\]

That is, \( \Delta r \) is approximately equal to \( v \cdot \Delta t \) for small angles \( \theta \). Properly speaking, the length we should use is the arc of the circle between \( p_1 \) and \( p_2 \) because that's the distance the point will actually be traveling. But for small angles, the difference between the length of the arc from \( p_1 \) and \( p_2 \) and the length of the chord from \( p_1 \) and \( p_2 \) can be ignored. Being able to ignore this will greatly simplify what follows.

If we substitute (5.11) into (5.10), we derive the acceleration of the point on the circle as follows:

\[
\frac{\Delta v}{v} = \frac{v \cdot \Delta t}{r},
\]

\[
\Delta v = \frac{v^2 \Delta t}{r},
\]

\[
\frac{\Delta v}{\Delta t} = \frac{v^2}{r}.
\]
The ratio $\Delta v/\Delta t$ is acceleration because it represents change in velocity over time. This is called *centripetal acceleration* because the direction of the bending force is always toward the center of the circle (see figure 5.5). It is defined as

$$a_c = \frac{v^2}{r}$$

*Centripetal Acceleration (5.12)*

where $a_c$ is centripetal acceleration, $v$ is velocity, and $t$ is time.

Suppose we can control a rocket in deep space and want it to turn in a circle around a point with radius $r$. To get it to turn, we would have to ignite one rocket on its tail to propel it forward with a force proportional to $v$ and ignite another pointing sideways with a force proportional to $a_c$. Figure 5.5a shows that for $v = 50$ and $r = 125$, $a_c$ must be $50^2/125 = 20$. Figure 5.5b shows that if $v$ is doubled to 100 for the same $r$, then $a_c$ must quadruple to 80 in order for the rocket to maintain a circle of the same size.

### 5.2.7 Tangential Velocity

On a merry-go-round the circular motion pushes riders away from the center, and pushes them harder, the further from the center they are. But is the direction of the push *radial*, directly away from the center? Setting an object on a turntable, we can spin it at some angular velocity $\omega$ sufficient to make it fly off. Suppose it flies off at point Q (figure 5.6). We would observe that the object’s angular velocity is instantly converted into some linear velocity in a direction *tangent* to the point where it flew off. This is understandable because

**Circular motion is linear velocity forward constrained by centripetal force toward a center.**

If we suddenly eliminate the centripetal force, the remaining linear velocity is all that is left, and the object flies off in whatever direction it was last aimed. In figure 5.6 the velocity of the
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Figure 5.6
Tangential speed.

Object at point Q is shown by a vector $v_T$ anchored on Q and drawn tangent to the circle. The vector $v_T$ indicates the **tangential velocity** of the object at point Q corresponding to its linear velocity.

Clearly, the object is subject to tangential velocity even when it is still on the turntable because this represents its linear velocity at each moment in time. Velocity implies both speed and direction, but the vector $v_T$ is constantly changing direction as it progresses around the circle. So the magnitude of the vector is just its length (without regard to which direction it points) and corresponds to its speed.

Intuitively, we can tell that the tangential speed $v_T$ must be related to the turntable's angular velocity $\omega = \theta/t$ as well as to its radius $r$ because an increase in either would tend to give more velocity to the object. But how can we express this?

Recall that (5.5) relates angular displacement $\theta$ and radius $r$ to the arc length $s$ by $s = r\theta$, and that (5.6) relates angular velocity to angular displacement and time by $\omega = \theta/t$. If we introduce (5.6) into (5.5), the result combines angular velocity and radius, as we require. Dividing both sides of (5.5) by time $t$, we obtain

$$
\frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t} \text{ rad/s.} \tag{5.13}
$$

The right-hand side now has a term $\theta/t$ in it. Since angular velocity $\omega = \theta/t$, (5.13) can be rewritten as

$$
\frac{s}{t} = r\omega \text{ rad/s.}
$$

Since $s$ measures arc length, the ratio $s/t$ expresses the speed of a point on the circle. Thus, *tangential speed* is defined as

$$
\frac{v_T}{t} = \frac{s}{t} = r\omega \text{ rad/s.}
$$

Tangential Speed \text{ (5.14)}
We must use units of rad/s because this equation was derived from (5.5), which defines radian measure. When an object is thrown off a turntable, its tangential speed is converted into tangential velocity because then it has a particular direction, namely, tangent to its last point of contact.

5.2.8 Period and Frequency

As the cone on the turntable in figure 5.1 completes one revolution, the corresponding simple harmonic motion completes one back-and-forth cycle. The period $T$ of this cycle clearly depends upon the angular velocity $\omega$ of the circle. Since by (5.6), $\omega = \theta/t$, and the circle completes one revolution of $\theta = 2\pi$ radians in $t = T$ seconds, we can relate angular velocity to period $T$ as follows:

$$\omega = \frac{\theta}{t} = \frac{2\pi}{T},$$

and so

$$T = \frac{2\pi}{\omega} \quad \text{(Period related to angular velocity)} \quad (5.15)$$

Since frequency $f = 1/T$, we can relate the angular velocity to frequency:

$$\omega = 2\pi \frac{1}{T} = 2\pi f.$$  

Relating angular velocity to frequency in this way will be so useful in subsequent chapters that it deserves being repeated:

$$\omega = 2\pi f. \quad \text{(Radian Velocity)} \quad (5.16)$$

In this book, when I write $\omega$, I will almost always mean its definition $2\pi f$. Solving (5.16) for $f$, we derive the definition of frequency:

$$f = \frac{\omega}{2\pi}. \quad \text{(Frequency related to angular velocity)} \quad (5.17)$$

This definition says that frequency is the ratio of the angular velocity, $\omega = \theta/t$ (see equation (5.6)), to the arc length of a circle. The greater the angular velocity, the more often it completes a full circle, hence the higher its frequency.

5.3 Projection of Circular Motion

Figure 5.7 shows a spring/mass system vibrating vertically next to a turntable that has a cone mounted on its edge. By appropriate choices of rotational speed of the turntable, elasticity of the spring, and weight of the mass, the motion of the shadows of the cone and mass can be synchronized on a screen behind them. This suggests that the simple harmonic motion of a pendulum or a weighted spring can be related to uniform circular motion via projection.
which defines radian angle into tangential point of contact.

ponding simple harmonic motion clearly depends upon the rate of one revolution as follows:

\[ \text{Angular velocity} \quad (5.15) \]

\[ \text{Solving} \quad (5.16) \quad \text{for} \quad f, \]

\[ \text{Angular velocity} \quad (5.17) \]

\( f \) (see equation (5.6)), completes a full cycle.

Table that has a cone base, elasticity of the mass can be synchronized of a pendulum or

\[ y = A \sin \theta = A \cdot \frac{y}{A}. \]

\[ \text{Sine Relation} \quad (5.18) \]

Figure 5.8 shows the turntable and screen from figure 5.7 with the cone at point Q. Since light shines across the circle parallel to the x-axis, point Q', which is the shadow of Q, appears on the screen at the same displacement above the x-axis. The displacement of points Q and Q' from the x-axis is y, the projection of the radius A onto the y-axis. Elementary trigonometry shows that the radius A, its angle \( \theta \), and the value of y are connected by the sine relation (see appendix A).

Equation (5.18) relates the height y of the triangle, and hence the height of its projection on the screen, to the radius A and its angle \( \theta \). The sine relation allows us to reconcile circular motion with
simple harmonic motion. In order to see how the vertical displacement $y$ changes, figure 5.9 adds a strip of film to record the position of the mass and cone through time, allowing us to see the sinusoidal motion of the spring/mass system together with the motion of the turntable. Mathematically and intuitively, it should be clear now that

*Simple harmonic motion and the projection of uniform circular motion are the same.*

### 5.3.1 Relating Displacement of Simple Harmonic Motion to Time

Since, by (5.6), $\theta = \omega t$, we can relate the vertical displacement $y$ of the cone’s shadow at time $t$ as follows:

$$y = A \sin \theta = A \sin \omega t,$$

(5.19)

where $A$ is the radius of the turntable, $\theta$ is the turntable’s angular displacement, $t$ is time, and $\omega$ is angular velocity.

The expression $\omega t$ in (5.19) determines the rotational position of the turntable at time $t$; taking the sine of that rotational position determines the height of the vertical displacement $y$; multiplying the vertical displacement by $A$ scales the displacement for the size of the turntable.

Equation (5.19) shows the identity of simple harmonic motion and circular motion and provides a way to determine the displacement of a sinusoidal wave at any time $t$. We see that

*The projection of simple harmonic motion through time generates sinusoidal motion.*

The term $A$ in (5.19) can be interpreted either as the radius of a circle or as the amplitude of the corresponding simple harmonic motion because this value determines both attributes.
5.4 Constructing a Sinusoid

A simple way to generate a sine wave is to plot a few selected points of (5.18) and connect the points with a smooth line. Figure 5.10 shows eight values of $A \sin \theta$ every 45° as $\theta$ makes one complete revolution. The y-axis shows the corresponding values of $A \cdot (y/A)$ for radius $A = 1$. A circle of radius 1 is a unit circle. It is convenient to set $A$ to 1 in order to keep the example simple, but it can be any value. Notice that $y$ takes on values in the range $-1$ to 1 as $\theta$ varies.

- When the angle $\theta$ is 0° or 180°, the displacement of $y = 0$ and $\sin 0^\circ = \sin 180^\circ = y/A = 0/1 = 0$.
- When $\theta$ is 90°, $y = 1$ and $\sin 90^\circ = y/A = 1/1 = 1$.
- When $\theta$ is 270°, $y = -1$ and $\sin 270^\circ = y/A = -1/1 = -1$.

These cardinal points are marked with diamonds in figure 5.10.

- At 45°, triangle $Axy$ in figure 5.8 becomes an isosceles right triangle, and by elementary geometry,
  $$y = \frac{A}{2} = \frac{A}{1.414}$$

  Plugging this value into the sine relation yields the formula
  $$\sin \frac{A}{\sqrt{2}} = \sin \frac{1}{\sqrt{2}} = \sin 45^\circ = 0.707 \ldots$$
5.4.1 Anatomy of a Sinusoid

The landmarks of the sinusoidal wave are shown in figure 5.11. The y-axis shows the amplitude, which is proportional to a corresponding circular radius \( A \). The x-axis shows the phases of the sine wave's various notable features, such as where it crosses the x-axis (zero crossings), and its crests and troughs. We speak of the “phases of the moon” in the same sense: phase describes the characteristic points reached periodically each time a wave repeats. The period, or cycle, of a sine wave is one complete movement through all its phases, corresponding to one complete revolution of a corresponding circle.

It will often be more convenient to show the passage of time on the x-axis rather than the size of the angle \( \theta \). Solving (5.7) for \( t \) yields

\[
t = \frac{\theta}{\omega},
\]

which shows that time is directly proportional to angular displacement \( \theta \). This means the x-axis can either measure elapsed time or elapsed phase.

Since frequency is the reciprocal of time, \( f = 1/t \), (5.20) can be rearranged:

\[
f = \frac{\omega}{2\pi},
\]

which shows that frequency is directly proportional to angular velocity \( \omega \). The greater the angular velocity, the more rapidly the turntable turns.

If the x-axis shows elapsed time, we are measuring frequency; if the x-axis shows elapsed phase, we are measuring periodicity.

Sinusoids, like circles, have no beginning and no end, so the period of a sine wave can start anywhere. Conventionally, sine wave periods are usually regarded as beginning at a positive-going
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zero crossing (figure 5.11) and extending until just before the next positive-going zero crossing. But we could just as well measure the period from crest to crest, or from trough to trough, by suitable choice of phase offset.

5.4.2 Phase Offset

Equation (5.19) requires that the turntable start at its 0° position, which is when point Q in figure 5.8 is aligned with its positive x-axis. In this position, the vertical displacement \( y = 0 \) because \( \sin 0 = 0 \). If we wish to be able to start the turntable at any orientation, we must introduce a way to specify its starting position in (5.19). If we don’t start with \( \theta = 0 \), \( y \) will have a nonzero initial value.

Let’s define a constant \( \phi \), which is the phase angle (or phase offset or phase shift) of the turntable’s starting position. The vertical displacement of the cone’s shadow at time \( t \) with phase offset \( \phi \) can then be written as

\[
y = A \sin(\omega t + \phi),
\]

(5.22)

where \( \phi \) defines a constant offset from 0°. It can take on any positive or negative real value. For instance, suppose we set \( \phi = \pi/2 \). Note in figure 5.10 that \( \sin(\pi/2) = 1 \). Then at time \( t = 0 \),

\[
y = A \sin(\omega t + \pi/2) = A \sin(\pi/2) = A.
\]

This means that at \( t = 0 \) the turntable starts rotating with the cone positioned at the top of the turntable, which is rotated 90° counterclockwise from the previous starting position.

5.4.3 Wavelength

The physical length of a waveform period, its wavelength, depends upon the medium through which the wave is traveling and its frequency. In air, sound waves travel at about 340 m/s (approximately 1100 feet per second) at a temperature of 20°C (see section 7.4).

So a frequency of 1 kHz in air has a wavelength of approximately

\[
\frac{1 \text{ second}}{1000 \text{ periods}} \cdot \frac{340 \text{ meters}}{1 \text{ second}} \approx 0.34 \text{ meters per period},
\]

or

\[
\frac{1 \text{ second}}{1000 \text{ periods}} \cdot \frac{1100 \text{ feet}}{1 \text{ second}} \approx 1.1 \text{ feet per period}.
\]

Note how these three measurements are interrelated.

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5.4.4 Velocity of Simple Harmonic Motion

How can we characterize the velocity of an object moving in simple harmonic motion when both the direction and speed of such an object change through time as the object vibrates back and forth? Since harmonic motion is the projection of circular motion, we should be able to understand the velocity of harmonic motion by thinking more about tangential velocity.

Figure 5.12 shows the projection of tangential velocity \( v_T \) of an object on the edge of a turntable. By a combination of geometry and trigonometry (see appendix A), we see that the velocity \( v \) of the shadow that is projected on the screen is just the \( y \)-axis component of the vector \( v_T \), that is,

\[
v = v_T \cos \theta, \tag{5.23}
\]

where \( \theta = \omega t \).

Recall from (5.14) that the tangential velocity \( v_T \) is related to the angular velocity \( \omega \) by \( v_T = r \omega \).

Let's substitute amplitude \( A \) for radius \( r \), so now \( v_T = A \omega \). Substituting \( A \omega \) for \( v_T \) in (5.23), we obtain the velocity of simple harmonic motion:

\[
v = A \omega \cos \theta = A \omega \cos \omega t. \tag{5.24}
\]

This tells us that even though an object on a rotating circle moves with uniform circular motion, the velocity of its corresponding simple harmonic motion is not uniform. The velocity constantly varies between maximum and minimum values through time sinusoidally. When \( \theta \) equals exactly 90° or 270°, velocity is exactly 0, and the object in simple harmonic motion is momentarily stationary. Velocity is positive maximum when \( \theta \) equals 0, and at that point it equals

\[
v = A \omega.
\]

Maximum Velocity of Simple Harmonic Motion \( (5.25) \)

Velocity is negative maximum when \( \theta \) equals 180°.
5.5 Energy of Waveforms

Equation (5.25) says that the velocity of an object vibrating in simple harmonic motion is proportional to both the amplitude and the angular velocity of the corresponding unit circle. In other words, simple harmonic motion—the projection of circular motion—will have higher velocity either if the corresponding circular motion has a longer radius or if that radius turns faster. This suggests that a mass moving in simple harmonic motion would have greater momentum if either its amplitude or its frequency were increased.

In section 4.14, kinetic energy $E_k$ was shown as the product of the mass $m$ of an object times the square of its velocity $v$, or $E_k = mv^2$. I used an automotive metaphor to show that doubling a car's speed quadruples its energy. Now let's apply this understanding to a molecule of air zipping in and out of someone's ear as part of a sound wave impinging on their eardrum.

If the amplitude of a wave doubles while the frequency remains the same, the particle must cover twice the distance in the same amount of time (via one period of doubled amplitude). Or, if the frequency of the wave doubles, the particle must cover twice the distance in the same amount of time (via two periods at the original amplitude). In either case, the energy of the molecule of air has quadrupled because the velocity of its simple harmonic motion has doubled.

If the wave in figure 5.13a is stretched out, it has the length shown in figure 5.13d. The wave in 5.13b, with twice the amplitude of the wave in 5.13a, has the length shown in 5.13e. Wave 5.13c has the same amplitude and twice the frequency of wave 5.13a, and its length also equals that shown in 5.13e. Since the duration $T$ of all three waves (5.13a, 5.13b, and 5.13c) is the same, but the length of waves 5.13b and 5.13c is twice that of wave 5.13a, clearly waves 5.13b and 5.13c have twice the speed of wave 5.13a. So we see that a wave's energy depends on both its amplitude and its frequency.

Consider a point on the turntable in figure 5.12. If the turntable's radius is $A$, it has circumference $d = 2\pi A$. Since, by equation (4.8), velocity is $v = \frac{dT}{dt}$, the circular velocity of the point is $v = 2\pi A/t$,

![Diagram](image)

**Figure 5.13**
Path lengths.
which also can be written as \( v = 2\pi f \cdot \theta \). Since, by equation (4.1), the frequency of rotation is \( f = \omega / 2\pi \), we can also write

\[
v = 2\pi f. \tag{5.26} \]

Taking \( E = m^2 \) from equation (4.28) and substituting \( v \) from (5.26) yields

\[
E = m(2\pi f)^2, \tag{5.27} \]

which confirms that wave energy depends upon both frequency and amplitude.

### 5.5.1 Measuring the Energy of Waveforms

**Peak Pressure Level** Perhaps the easiest way to measure the strength of a waveform is to examine how its *maxima* and *minima*—its highest and lowest points—relate to the ambient pressure level. *Peak pressure level* of a sound wave is the difference between the ambient pressure level and the magnitude of either the maximum or minimum pressure level of the sound wave, whichever is greater:

\[
l_p = \max(|l_\text{p}|, |l_\text{a}|) - l_\text{a}, \tag{5.28} \]

where \( l_p \) is peak pressure level, \( l_a \) is ambient pressure level, \( l_p \) is the highest peak, and \( l_a \) is the deepest trough. The operator \( \max \) gives the magnitude of the enclosed expression, and the function \( \max(a, b, \ldots) \) chooses the greatest value of its arguments. Figure 5.14 shows the peak pressure level.

**Peak-to-Peak Pressure Level** Every sound recording device has some limit beyond which it can no longer accurately represent the strength of the waveform being recorded, and waveforms with peaks greater than the limit are distorted (see section 4.24.2). Modern recorders often contain volume level meters that measure the strength of the recorded waveform based on (5.28) to help the recordist prevent distortion. The peak-to-peak pressure level of a waveform is the magnitude of the distance between \( l_p \) and \( l_a \):

\[
l_{pp} = |l_p - l_a| \tag{5.29} \]

**Why Average Pressure Level Doesn’t Work** Peak-to-peak level shows the limits of a waveform’s amplitude, but it does not always provide the best information about a waveform’s strength. For example, a recording that is mostly silence except for a brief tone burst may have a large peak

![Figure 5.14](image)

*Figure 5.14*  
Peak pressure level.
amplitude if the tone burst is loud, but there is little energy in the waveform over its total duration because it is mostly silent.

One might try to get a clear picture of a waveform’s strength by averaging the waveform’s pressure over time, hoping to smooth out the peaks. But sound waveforms are usually evenly balanced above and below ambient pressure, so in general \( l_+ - l_- \equiv 0 \). Therefore, the mean value of most sounds is typically close to zero, and so average pressure is not a useful way to measure the strength of a waveform.

**RMS Level** Ideally, it would be useful to observe the power contained in the waveform because power is the energy in the waveform through time. But all we can easily measure with a microphone is the waveform’s pressure fluctuations. How can we derive a measure of energy from pressure? The key lies in recalling that there is a square relation between amplitude and energy.

The average value of \( \cos t \) over one full period is 0.0. The peak amplitude \( l_p = |l_+| = |l_-| \) of the cosine is 1.0. The peak-to-peak amplitude is \( l_{pp} = 2.0 \) (figure 5.15).

Let \( s(t) = \cos t \). There is a trigonometric identity (see volume 2, appendix A.4.1) that says

\[
\cos a \cos b = \frac{\cos(a-b) + \cos(a+b)}{2}
\]

If we square \( s(t) \), then

\[
s^2(t) = \cos^2 t = \cos t \cos t = \frac{\cos(t-t) + \cos(t+t)}{2} = \frac{1 + \cos 2t}{2}
\]

(5.30)

So, by (5.30), \( s^2(t) \) is a cosine wave at twice the frequency, offset by 1, and then divided by 2 (figure 5.16). This is what the original cosine waveform, shown in figure 5.15, looks like when squared. Note that all values are now positive. The peak value is still 1.0. Its *mean value* is 0.5.

Now let’s take the mean value for this squared waveform (0.5) and undo the effects of the squaring operation. The square root of the mean value is \( 0.5^{1/2} \approx 0.707 \). This is the root mean squared

![Figure 5.15](image)

**RMS level.**
(RMS) value of the waveform. So the RMS amplitude of \( s(t) \equiv 0.707 \). This allows us to say something useful about the average energy of a sinusoid knowing only its amplitude. The relation of the amplitudes is as follows:

- **Average**: 0
- **RMS**: 0.71
- **Peak**: 1
- **Peak-to-peak**: 2

In general, if \( s(t) \) is a sinusoid with peak amplitude \( A \), then its RMS amplitude is \( A/\sqrt{2} \).

Because we used a sinusoid to derive RMS amplitude, this measure is only valid for sinusoids. In particular, it is not valid for time-varying waveforms. This, of course, leaves out all the interesting real-world audio waveforms we'd like to measure with it. Nonetheless, this definition of RMS is widely used in practice because, I suppose, it's better than nothing. But there are more sophisticated techniques to overcome this difficulty and find the true RMS value of arbitrary waveforms (see volume 2, chapter 1).

**Sound Pressure Level**  Although the decibel scale was developed for sound intensity, we can adapt it to measure sound pressure level. Equation (4.40) defined decibels of sound intensity level (dB SIL) as

\[
y \text{ dB SIL} = 10 \log_{10} \frac{I'}{I}, \quad \text{dB SIL} \quad (5.31)
\]

where \( I \) is a reference intensity, and \( I' \) is the intensity being measured. Recalling that intensity is proportional to the square of amplitude, we can define decibels of *sound pressure level (dB SPL)* as

\[
10 \cdot \log_{10} \left( \frac{A'}{A} \right)^2
\]

\[
2 \cdot 10 \cdot \log_{10} \frac{A'}{A}
\]

and

\[
y \text{ dB SPL} = 20 \log_{10} \frac{A'}{A}, \quad \text{dB SPL} \quad (5.32)
\]

where \( A \) is a reference amplitude, and \( A' \) is the amplitude being measured.
Decibels of sound pressure level (SPL) correspond to twice the equivalent decibels of sound intensity level (SIL). Where a doubling of intensity corresponds to an increase of 3 dB SIL, a doubling of pressure corresponds to an increase of 6 dB SPL. An intensity ratio of 10:1 equals 10 dB SIL and 20 dB SPL.

5.6 Summary

Uniform circular motion is circular movement at a constant speed. Simple harmonic motion is the projection of circular motion. Angular displacement is the angle through which a rigid body rotates about its axis of rotation. Counterclockwise angular displacement is taken to be positive and clockwise angular displacement to be negative. The angle formed by a radius and an arc the length of the radius is called a radian. Measuring angles with radians simplifies many calculations. Angular velocity is the rate at which angular displacement changes. Angular acceleration is the rate at which angular velocity changes.

By Newton's laws, objects tend to travel in a straight line. To travel in a circular path, an object must experience centripetal acceleration to overcome the object's tendency to travel linearly. Circular motion is linear velocity forward constrained by centripetal force toward a center. There is no such thing as centrifugal force.

Simple harmonic motion of a pendulum or a weighted spring can be related to uniform circular motion via projection. Simple harmonic motion is the same as the projection of uniform circular motion. The projection of simple harmonic motion through time generates sinusoidal motion.

An object on a rotating circle moves with uniform circular motion, but the velocity of its corresponding simple harmonic motion constantly varies between maximum and minimum values through time sinusoidally. The speed of an object vibrating in simple harmonic motion is proportional to both the amplitude and the angular velocity of the corresponding unit circle.

Peak pressure level of a sound wave is the difference between the ambient pressure level and the magnitude of either the maximum or minimum pressure level of the sound wave, whichever is greater. The peak-to-peak pressure level of a waveform is the magnitude of the distance between its lowest and highest point. The root mean squared (RMS) value of a waveform is a useful measure of energy in a sinusoid, calculated by squaring the waveform to derive its mean value and then taking the square root of the mean value to determine the RMS value. Technically, this operation is valid only for sinusoids.

Since it's easier to measure pressure variations in air than sound intensity, we adapt the decibel of sound intensity level (dB SIL) to the decibel of sound pressure (dB SPL) by doubling the dB SIL value.