This is the Appendix of [Cve13]. The full text of [Cve13] is available online or send a request to andrej@bu.edu. See also [CC12, CC13, CC11].

Appendix A

Elements

A.1 Möbius transformations

Möbius transformations are a class of transformations of the complex plane that preserve generalized circles. The special Möbius transformations that take $\mathbb{D}$ to $\mathbb{D}$ and preserve the hyperbolic distance have the form

$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}},\ \ a,b\in\mathbb{C},\ \ |a|^2-|b|^2\neq 0. \tag{A.1}$$

A.2 The matrix of composition $(f\circ g)(z)$ from matrices F and G

Write $f\leftrightarrow F$ if $f(z)=\dfrac{az+b}{cz+d}$ corresponds to $F=\begin{bmatrix} a & b\\ c & d\end{bmatrix}$

If $f(z):=\dfrac{a_1z+b_1}{c_1z+d_1}\leftrightarrow F=\begin{bmatrix} a_1 & b_1\\ c_1 & d_1\end{bmatrix}$ and

$g(z):=\dfrac{a_2z+b_2}{c_2z+d_2}\leftrightarrow G=\begin{bmatrix} a_2 & b_2\\ c_2 & d_2\end{bmatrix}$

then $(f\circ g)(z)\leftrightarrow F\cdot G=\begin{bmatrix} a_1a_2+b_1c_2 & a_1b_2+b_1d_2\\ a_2c_1+c_2d_1 & b_2c_1+d_1d_2\end{bmatrix}$.

Namely: $(f\circ g)(z)=f(g(z))=\frac{z(a_1a_2+b_1c_2)+a_1b_2+b_1d_2}{z(a_2c_1+c_2d_1)+b_2c_1+d_1d_2}$.

In particular, if f is represented by F, then
$f^i(z)\overset{\Delta}{=}\underbrace{(f\circ f\circ\ldots f)}_{i\ \text{times}}(z)$ corresponds to $F^i=\prod_1^i F$.

A.3 The matrix of $f^{-1}(z)$ given F

If $f \leftrightarrow F$ and $g \leftrightarrow G$ and $g = f^{-1}$,

then $f^{-1} \leftrightarrow -|F|F^{-1} = G$ where $|F|$ is the determinant of F.

Also, by symmetry, $F = -|G|G^{-1}$, and $|F| = |G|$

Explicitly, if $F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $G = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$.

A.4 Fractional conformal mapping

A fractional conformal mapping is determined by three points and their images under the mapping. Given three points and the corresponding images under $w = w(z) = \dfrac{az+b}{cz+d}$, here we find the coefficients a, b, c, d.

Let $w_k = w(z_k)\,,\ k = 1,2,3$. Then

$$w_i - w_j = \frac{az_i + b}{cz_i + d} - \frac{az_j + b}{cz_j + d} = \frac{(z_i - z_j)\,|W|}{(cz_i + d)\left(cz_j + d\right)}$$

where $W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $|W|$ is the determinant of W. Observing the ij invariance of the denominator, we have

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

Solving i.t.o. $w = w(z)$,

$$w(z) = \frac{-\left(w_1\left(w_2(z - z_3)(z_1 - z_2) + w_3(z - z_2)(z_3 - z_1)\right) + w_2 w_3 (z - z_1)(z_2 - z_3)\right)}{w_1(z - z_1)(z_2 - z_3) + w_2(z - z_2)(z_3 - z_1) + w_3(z - z_3)(z_1 - z_2)}.$$

That is,

$$a = w_2 w_1 (z_2 - z_1) + w_1 w_3 (z_1 - z_3) + w_3 w_2 (z_3 - z_2)$$
$$b = w_1 w_2 z_3 (z_1 - z_2) + w_1 w_3 z_2 (z_3 - z_1) + w_2 w_3 z_1 (z_2 - z_3)$$
$$c = w_1 (z_2 - z_3) + w_2 (z_3 - z_1) + w_3 (z_1 - z_2)$$

$$d = w_1 z_1 (z_3 - z_2) + w_2 z_2 (z_1 - z_3) + w_3 z_3 (z_2 - z_1)$$

□

A.5 The invariant point of a transform

Let

$$f(z) = \frac{az+b}{cz+d}.$$

The invariant point of the transform is the solution of $z = f(z)$, that is

$$z = \frac{a - d \pm \sqrt{a^2 - 2ad + 4bc + d^2}}{2c}.$$

A.6 Geodesic through 2 points

In the context of the Poincaré disk model, given two points $A(x_a, y_a)$ and $B(x_b, y_b)$ in or on the unit circle, we can find the coordinates of the center $C(x_c, y_c)$ and the radius R of the geodesic through A and B. This can be completed using suitably chosen Möbius transformations, but here we opt to proceed without an appeal to them. All coordinates x_i, y_i herein are Euclidean.

As in Figure A.1, let the Euclidean bisector of the segment AB be be $y = ax + b$. The slope of AB is $s_{AB} = (y_b - y_a) / (x_b - x_a)$ and the slope of bisector is thus

$$a = -1/s_{AB} = -(x_b - x_a) / (y_b - y_a)$$

The bisector passes through the midpoint of AB $M(x_m, y_m)$ with $x_m = (x_a + x_b)/2$. $y_m = (y_a + y_b)/2$ and satisfies $y_m = ax_m + b$ whence

$$b = y_m - ax_m.$$

Further, $R^2 + r^2 = \overline{OC}^2$ with $r = 1$ since $\triangle OCD$ is a right-angled triangle. Also, $R = \overline{AC} = \overline{BC}$ since C is on the bisector of AB. Therefore, $\overline{AC}^2 + 1 = \overline{OC}^2$. Substituting $\overline{AC}^2 = (x_c - x_a)^2 + (y_c - y_a)^2$ and $\overline{OC}^2 = x_c^2 + y_c^2$ we have $(x_c - x_a)^2 + (y_c - y_a)^2 + 1 = x_c^2 + y_c^2 \Rightarrow$

$$-2x_c x_a + x_a^2 - 2y_c y_a + y_a^2 + 1 = 0.$$

Substituting $y_c = ax_c + b$, yields

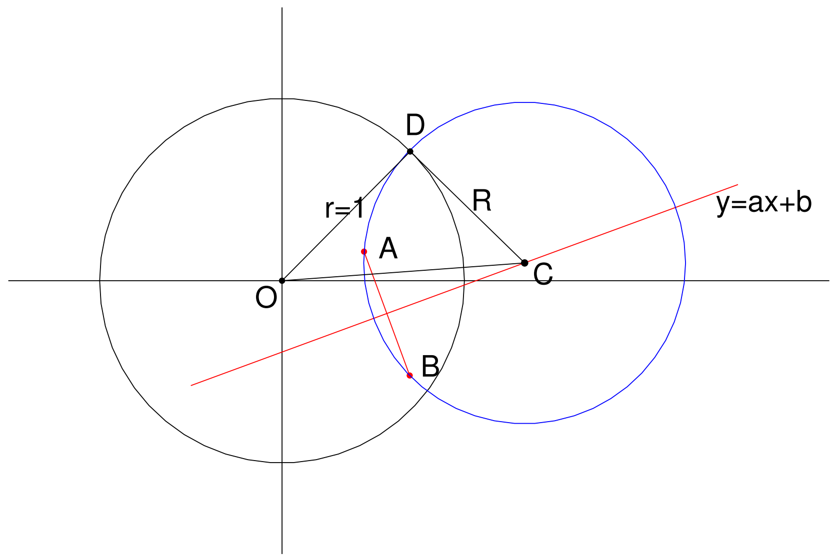

Figure A.1: Geodesic through 2 points

$$-2x_c x_a + x_a^2 - 2\left(ax_c + b\right) y_a + y_a^2 + 1 = 0$$

whence

$$x_c = \frac{{x_a}^2 + {y_a}^2 + 1 - 2by_a}{2\left(ay_a + x_a\right)}$$

and we can calculate

$$y_c = ax_c + b$$

$$R = \sqrt{\left(y_c - y_a\right)^2 + \left(x_c - x_a\right)^2}$$

A.7 Geodesic through 2 ideal points

In the context of the Poincaré disk model, given two ideal points a and b on the unit circle, we can find the center c and the radius R of the geodesic through a and b. The derivation shown here does not make use of Möbius transformations. All coordinates are Euclidean complex.

In Figure A.2, $\triangle ABC$ is a right-angled triangle and $b = \sqrt{pq}$. Namely, by similarity of triangles, $p/b = b/q$ and $b^2 = pq$.

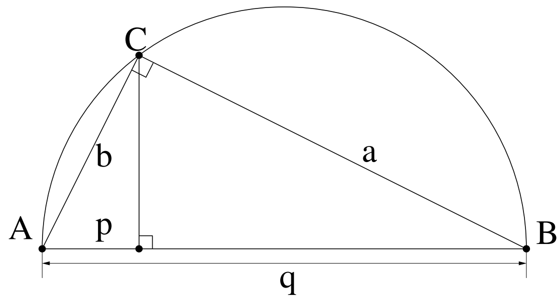

Figure A.2: Geometric mean

In Figure A.3, a and b are ideal points on the unit circle, G is a geodesic with center c and radius R, and $m = (a+b)/2$ is the midpoint of the Euclidean segment ab. $r = 1$ is the radius of the unit circle.

We have $r = \sqrt{pq} \Rightarrow 1 = pq \Rightarrow q = 1/p = 1/|m|$. The center of the geodesic c is

$$c = \frac{m}{|m|} \cdot q = \frac{m}{|m|\,|m|} = \frac{m}{|m|^2} = \frac{m}{m \cdot \overline{m}} = \frac{1}{\overline{m}} = \frac{2}{\overline{a} + \overline{b}} \tag{A.2}$$

where $\overline{m}$ is the complex conjugate of m. The radius can be subsequently calculated as

$$R = |c - a|\,. \tag{A.3}$$

When a and b are on a diameter of the unit circle, we have $m = 0$ and $c = 1/\overline{m} \to \infty$ and $R \to \infty$. Thus Equations (A.2) and (A.3) hold for this case as well.

We note that

1. the midpoint m of the Euclidean segment ab is the reflection of the origin O in the geodesic defined by a and b Namely, $R = \sqrt{\overline{OC} \cdot \overline{mC}} \Rightarrow R^2 = \overline{OC} \cdot \overline{mC}$.
2. The center c and the midpoint m are reflections of each other in the unit circle. Namely, $pq = 1 = r^2$.

A.8 Reflection of a point from a geodesic

Given a geodesic in the Poincaré disk model and a point P we can find the reflection Q from the geodesic.

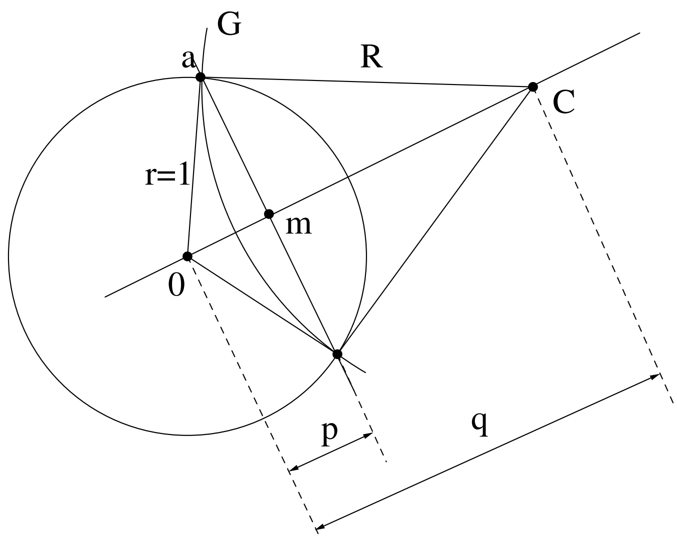

Figure A.3: Geodesic through 2 ideal points

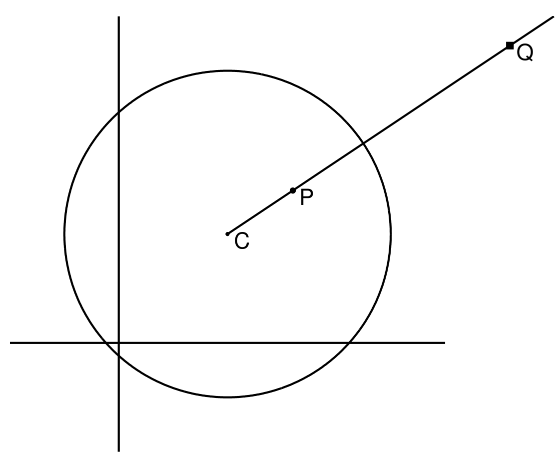

Figure A.4: Reflection of a point from a geodesic

Generally, given a circle $K(C,R)$ with center $C(x_c,y_c)$ in Euclidean coordinates and radius R and points $P(x_p,y_p)$ and $Q(x_q,y_q)$ on the ray CP, then we say P and Q are reflections of each other in the circle K if $\overline{PC}\cdot\overline{QC}=R^2$. P can be on the circle in which case $P\equiv Q$. If $P\equiv C$ then Q is at ∞. To find the coordinates of Q given the coordinates of P,

$$x_q = x_p + \Delta\cos\alpha \tag{A.4}$$

$$y_q = y_p + \Delta\sin\alpha \tag{A.5}$$

where

$$\Delta = \rho_q - \rho_p \tag{A.6}$$

$$\rho_p = \sqrt{(x_p - x_c)^2 + (y_p - y_c)^2} \tag{A.7}$$

$$\rho_q = \frac{R^2}{\rho_p} \tag{A.8}$$

$$\alpha = \frac{\pi}{2}\operatorname{sign}(y_p - y_c) - \tan^{-1}\frac{x_p - x_c}{y_p - y_c} \tag{A.9}$$

The above, in complex coordinates: $z_p = x_p + iy_p$, $z_c = x_c + iy_c$, $z_q = x_q + iy_q$

$$z_q = z_p + \Delta e^{i\alpha}$$

where

$$\begin{aligned} \Delta &= \rho_q - \rho_p & \rho_p &= \left|z_p - z_c\right| \\ \rho_q &= R^2/\rho_p & \alpha &= \angle(z_p - z_c) \end{aligned}$$

Listing A.1: `refl()`

```
function zq=refl(zp,zc,R)

ro_p=abs(zp−zc);
ro_q=R^2./ro_p;
zp=zp+i∗(imag(zp−zc)==0)/10000;
angl=pi/2∗sign(imag(zp−zc))−atan(real(zp−zc)./imag(zp−zc));
zq=zp+(ro_q−ro_p).∗exp(i∗angl);
```

Eqs. (A.4)–(A.9) work without modification for P outside of the circle. Eq. (A.9) returns the angle $\angle(CP,Ox)$ between the ray CP and the positive x-axis: $\alpha\in[-\pi,\pi)$, unlike

$\tan^{-1}\left(\frac{y}{x}\right)$ which returns angles only in quadrants 1 and 4.

Listing A.1 shows `refl()`, a Matlab implementation of Eqs. (A.4)–(A.9). `refl()` can take a vector z_p of complex points and returns the vector of corresponding images. Line 4 in Listing A.1 is to avoid division by zero in (A.9) by replacing zeros with 10^{-4}.

A.9 Hyperbolic distance in the Poincaré disk

The advantage of using the Poincaré disk model $\mathbb{D}$ over the half-plane model of the hyperbolic plane is that there exists an explicit formula to convert between hyperbolic and Euclidean distance for a pair of points in $\mathbb{D}$.

The formula that links Euclidean and hyperbolic distance between z_1 and z_2 in $\mathbb{D}$ is

$$\frac{|z_1 - z_2|^2}{\left(1-|z_1|^2\right)\left(1-|z_2|^2\right)} = \frac{1}{2}(\cosh d - 1) = \sinh^2 \frac{d}{2} \tag{A.10}$$

where d is the hyperbolic distance $d_{\mathbb{D}}(z_1, z_2)$ and $|z_1 - z_2|$ is the Euclidean distance. The proof is by direct calculation [And07, Prop4.3]

Solving cosh:

$$\cosh a = \frac{e^a + e^{-a}}{2} = x$$

$$(e^a)^2 - 2xe^a + 1$$

$$e^a = x + \sqrt{x^2-1} \qquad a = \log\left(x \pm \sqrt{x^2-1}\right)$$

But also

$$\left(x+\sqrt{x^2-1}\right)\left(x-\sqrt{x^2-1}\right) = 1$$

$$\log\left(x+\sqrt{x^2-1}\right) + \log\left(x-\sqrt{x^2-1}\right) = 0$$

so only one is positive and finally

$$a = \log\left(x+\sqrt{x^2-1}\right).$$

$\square$

As an alternative to Equation (A.10), the hyperbolic distance between x and y in $\mathbb{D}$ can be calculated using

$$d(x,y) = 2\text{atanh}\left|\frac{y-x}{1-\bar{x}y}\right| \tag{A.11}$$

Namely, for $x \neq y$ points in the Poincaré disk $\mathbb{D}$, choose

$$p(z) = \frac{az+b}{\bar{b}z+\bar{a}}$$

with $|a|^2 - |b|^2 = 1$. $p(z)$ moves the pair (x,y) to $(0, p(y))$ with $p(y)$ real and positive:

$$p(z) = \frac{a(z-x)}{\bar{a}(-\bar{x}z+1)};$$

Then

$$p(x) = 0,\ p(y) = \frac{a(y-x)}{\bar{a}(1-\bar{x}y)} > 0$$

and

$$d(x,y) \quad = \quad d(0, p(y)).$$

To find the hyperbolic length of a segment $\overline{OA}$ where O is the center and A is any point on the positive real line such that $\overline{OA}$ has Euclidean length r:

Parametrize in Euclidean rectangular coordinates: $f(t) = t$, $t \in [0,r]$. (That is, $x(t) = t$, $y(t) = 0$). Then $|z| = |t| = t$ in the given interval and $|dz| = |f'(t)|\,dt = |t'|\,dt = dt$

$$\begin{aligned}
\ell_D f &= \int_f \frac{2|dz|}{1-|z|^2} = \int_{t=0}^{t=r} \frac{2dt}{1-t^2} = \\
&= \int_{t=0}^{t=r} \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt = \\
&= \ln\left(\frac{1+r}{1-r}\right) = 2\tanh^{-1}(r).
\end{aligned}$$

$$d(0, p(y)) = 2\text{atanh}(p(y)) = 2\text{atanh}\left(\frac{a(y-x)}{\bar{a}(1-\bar{x}y)}\right).$$

But since $p(y) > 0$,

$$p(y) = |p(y)|,$$

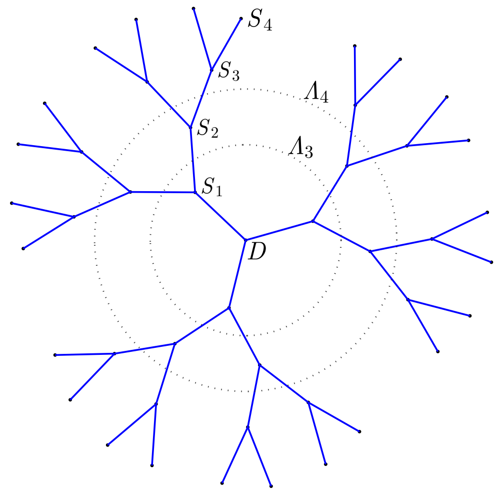

Figure A.5: A simplified model

so

$$d(0, p(y)) = 2\text{atanh}(p(y)) = 2\text{atanh}\left|\frac{y-x}{1-\bar{x}y}\right| = d(x,y). \tag{A.12}$$

□

Eq. (A.12) is computationally more convenient than (A.10).

A.10 Number of next hop candidates

In the context of the analysis of Section **??**, consider a rooted d-regular tree. For example, Figure A.5 shows such a tree with $d = 3$ (binary tree). Let $d_1 = d - 1$ be the number of children at each non-leaf node. Take the root node D to be at level $\ell = 1$, its children nodes at level $\ell = 2$, etc. We have for the total number of nodes up to level L:

$$\begin{aligned} n &= 1 + d + d(d-1) + d(d-1)^2 + \ldots + d(d-1)^{L-2} = \\ &= \frac{d(d-1)^{L-1} - 2}{d-2} = \frac{(d_1+1)\cdot d_1^{L-1} - 2}{d_1 - 1}. \end{aligned}$$

Therefore the exact number of nodes in Λ_ℓ is

$$\#\Lambda_\ell = \frac{(d_1+1)\cdot d_1^{\ell-2} - 2}{d_1 - 1}$$

and

$$p_\ell = \frac{\#\Lambda_\ell}{n} = \frac{(d_1+1)\, d_1^{\ell-2} - 2}{(d_1+1)\, d_1^{L-1} - 2}.$$

The free terms can be easily omitted for the d values typically occurring in the graphs of interest in this work, whence

$$p_\ell \approx \frac{d_1^{\ell-2}}{d_1^{L-1}} = d_1^{-(L-\ell+1)}.$$

Appendix B

K Embedding

This section outlines the details of the implementation of the d-regular tree greedy embedding procedure [Kle07], used in Chapters **??** and **??**.

As in [Kle07], every node w in the infinite regular tree of rooted at node r has an associated Möbius transformation $\mu_w()$ such that the node's complex coordinate in the Poincaré disk model greedy embedding of the tree can be calculated as

$$C_w = \mu_w^{-1}(v) \tag{B.1}$$

where

$$v = -\sigma(0) \tag{B.2}$$

is a constant that can be calculated for a given d (see below).

For a node w in the regular tree of degree d, let the $w_0, w_1, \ldots, w_{d-1}$ be a relative naming of w's direct neighbors such that w_0 is the parent node of w relative to the root r. Let w's Möbius transformation $\mu_w()$ have a corresponding matrix M_w (as in Section A.3). Then for $k = 1..d-1$ the functions $\mu_{w_k}()$ of the neighbors w_k of w have corresponding matrices given by

$$M_{w_k} = B^k \cdot A \cdot M_w \tag{B.3}$$

and their complex coordinates in the Poincaré disk model are given by $C_{w_k} = \mu_{w_k}^{-1}(v)$.

The matrix $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ does not depend on d and its corresponding transformation is $a(z) = -z$. The matrix B is given by

$$B = SRS^{-1} \tag{B.4}$$

where

$$R = \begin{bmatrix} e^{i2\pi/d} & 0 \\ 0 & 1 \end{bmatrix}$$

since $\rho(z) \leftrightarrow R$ is the rotation $\rho(z) = ze^{i2\pi/d} = \left(ze^{i2\pi/d} + 0\right)/(0z+1)$.

To obtain the matrix S corresponding to $\sigma(z)$, we apply the method given in Section A.4 to the points

$z_1 = 1$	$w_1 = 1$
$z_2 = z_P$	$w_2 = 0$
$z_3 = z_B$	$w_3 = -1$

Refer to Figure B.1 for the calculation of z_P and z_B. In the figure, C is the Euclidean center of the geodesic between the ideal points A and B defining one side of the ideal polygon with vertices $e^{ik2\pi/d}$, $k = 0..d-1$. The complex coordinate of C is $z_C = x_C + iy_C$. Clearly $x_C = 1$ and from $\triangle OAC$ we have $y_C = -\tan\frac{\pi}{d}$; the Euclidean radius of the arc AB is $R = |y_c| = \tan\frac{\pi}{d}$. Let P be the intersection of the ray OC and the arc APB. Clearly P is the center of the arc APC. Let Δ denote the Euclidean distance of the segment OC. Then $\Delta = \overline{OC} - R = \sqrt{x_C^2 + y_C^2} - R = \sqrt{1+R^2} - R$. and the coordinates of P are

$$\begin{aligned} x_P &= \Delta\cos\frac{\pi}{d} \\ y_P &= -\Delta\sin\frac{\pi}{d} \end{aligned}$$

whence $z_P = x_P + iy_P = \Delta e^{-i\pi/d}$. On the other hand, $z_B = e^{-i2\pi/d}$.

Alternatively, as in [Kle07], the midpoint of the arc AQB, z_Q, given by $z_Q = e^{-\frac{\pi}{d}} = z_B^{1/2}$ can be used to calculate $\sigma()$ in which case the points to be used with the Section A.4 method are

$z_1 = 1$	$w_1 = 1$
$z_2 = {z_B}^{1/2}$	$w_2 = -i$
$z_3 = z_B$	$w_3 = -1$

In any case we would have

$$\sigma(z) = 1 - \frac{z\,(2+2\mathrm{i}) + \mathrm{e}^{\frac{\pi \mathrm{i}}{d}}\,(2+2\mathrm{i}) - z\mathrm{e}^{\frac{\pi \mathrm{i}}{d}}\,(2+2\mathrm{i}) - 2 - 2\mathrm{i}}{2\cos\!\left(\frac{\pi}{d}\right) - 2\sin\!\left(\frac{\pi}{d}\right) - 2z\mathrm{e}^{\frac{\pi \mathrm{i}}{d}} - 2 + z\mathrm{e}^{\frac{\pi 2\mathrm{i}}{d}}\,(1-\mathrm{i}) + z\,(1+\mathrm{i})}.$$

The calculation of the node coordinates according to Eqs. (B.1) and (B.3) would require the inversions

$$M_{w_k}^{-1} = M_w^{-1} \cdot A^{-1} \cdot (B^{-1})^k. \tag{B.5}$$

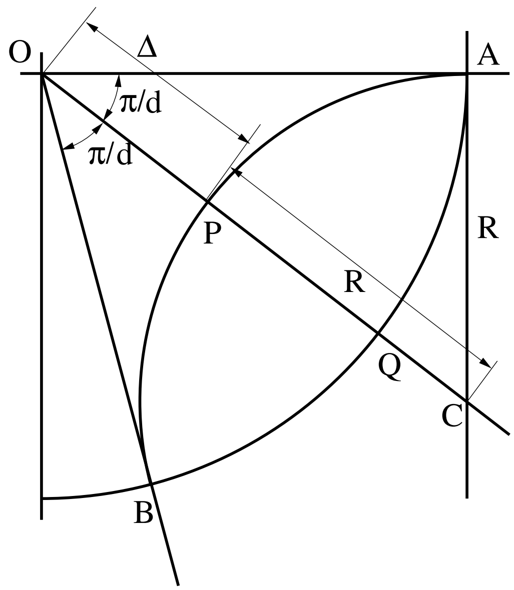

Figure B.1: z_P and z_B

It is interesting to note that B^{-1} has a more compact symbolic expression compared to B. It can be obtained by substituting S and R into Eq. (B.4):

$$B^{-1} = SR^{-1}S^{-1} = \begin{pmatrix} \frac{1}{2}\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}}+1\right)-\mathrm{i}\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}} & -\frac{1}{2}\left(1+\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}}\right) \\ -\frac{1}{2}\left(1+\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}}\right) & \frac{1}{2}\left(\mathrm{e}^{-\frac{2\pi\mathrm{i}}{d}}+1\right)+\mathrm{i}\mathrm{e}^{-\frac{\pi\mathrm{i}}{d}} \end{pmatrix} \tag{B.6}$$

Listing B.1 contains a Matlab calculation of B^{-1} for a given value of d in variable-precision arithmetic (VPA).

Listing B.1: `K Embedding Calculations`

```
RR=tan(vpa(pi)/d);
DEL=sqrt(1+RR^2)-RR;
zP=DEL*exp(-i*vpa(pi)/d);
zB=exp(-i*2*vpa(pi)/d);
z1=1 ; w1=1 ; z2=zP ; w2=0 ; z3=zB ; w3=-1 ;
wa=-w1*w2*(z1-z2)-w1*w3*(z3-z1)-w2*w3*(z2-z3);
wb=w1*w2*(z1-z2)*z3 + w1*w3*(z3-z1)*z2 + w2*w3*(z2-z3)*z1;
wc=w1*(z2-z3) +w2*(z3-z1) + w3*(z1-z2);
wd=-w1*(z2-z3)*z1 -w2*(z3-z1)*z2 -w3*(z1-z2)*z3;

S=[wa wb ;wc wd];
% A=[ a b; c d ] = [A(1) A(3) ; A(2) A(4)]
v=-S(3)/S(4);
```

```
% B^-1
B_1 = @(d)[1./(2.*exp((2.*vpa(pi).*i)./d))-i./exp((vpa(pi).*i)./d)+vpa(1)./2
      -1./(2.*exp((2.*vpa(pi).*i)./d))-vpa(1)./2;
-1./(2.*exp((2.*vpa(pi).*i)./d))-vpa(1)./2 (exp((2.*vpa(pi).*i)./d)+2.*i.*exp((
      vpa(pi).*i)./d)+vpa(1))./(2.*exp((2.*vpa(pi).*i)./d))];

B_1d=B_1(d); % substitute the concrete value of d

A=vpa([-1 0; 0 1]);
A_1=A^-1;
```

We note that the fixed point $\sigma(0)$ in Eq. (B.2) (see also Section A.5) and the matrix B with unit determinant in the cases $d = 3, 4, 6$ have simple, closed-form symbolic expressions summarized below.

d	$\sigma(0)$	B
3	$i\left(2-\sqrt{3}\right)$	$\begin{bmatrix} 1/2+i & 1/2 \\ 1/2 & 1/2-i \end{bmatrix}$
4	$i\left(\sqrt{2}-1\right)$	$\begin{bmatrix} \sqrt{2}/2+i & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2-i \end{bmatrix}$
6	$i\frac{\sqrt{3}}{3}$	$\begin{bmatrix} \sqrt{3}/2+i & \sqrt{3}/2 \\ \sqrt{3}/2 & \sqrt{3}/2-i \end{bmatrix}$

Appendix C

C Embedding

Listing C.1 contains a Matlab implementation of the C Embedding [CC09].

Listing C.1: C Embedding

```
function coord=f(predecessors,root,list)

% 1. INITIALIZE
alpha(root)=pi; % real
beta(root)=2*pi; % real

aa(root)=exp(i*alpha(root)); % complex
bb(root)=exp(i*beta(root)); % complex

geo_ctr(root)=2/( conj( aa(root) + bb(root) ) ); % complex
geo_rad(root)=abs(geo_ctr(root) - aa(root)); % real

coord(root)=-0.1-i*0.1; % complex

% 2. FOR EACH
for curnode=list(2:end),

    parent=predecessors(curnode);

    % 2a (i)
    alpha(curnode) = alpha(parent);
    beta (curnode) = ( alpha(parent)+beta(parent) ) / 2; % real

    % 2a (ii)
    alpha(parent) = beta(curnode); % update

    % 2b (i)
    aa(curnode) = exp(i*alpha(curnode)); % complex
    bb(curnode) = exp(i*beta(curnode)); % complex
    geo_ctr(curnode) = 2 / ( conj( aa(curnode) + bb(curnode)) ); % complex
```

```
    geo_rad(curnode)=abs(geo_ctr(curnode)-aa(curnode)); % complex

    coord(curnode)=geo_rad(curnode)^2/(conj(coord(parent)-geo_ctr(curnode)
        ))+geo_ctr(curnode); % complex

    % 2b (ii)
    alpha(curnode)= ( alpha(curnode) + beta(curnode) ) / 2;

end
```

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