

# Zermelo-Fraenkel Axioms, Internal Classes, External Sets

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## Abstract

Usual math sets have special types: countable, compact, open, occasionally Borel, rarely projective, etc. They have finite “depth”: are described by a single Set Theory formula with parameters unrelated to other formulas. Exotic expressions referring to sets with no depth limit or to Powerset axiom appear mostly in esoteric or foundational studies.

Recognizing internal to math (formula-specified) and external (based on parameters in those formulas) aspects of math objects greatly simplifies foundations. I postulate external sets (not internally specified, treated as the domain of variables) to be hereditarily countable and independent of formula-defined classes, i.e. with finite Kolmogorov Information about them. This allows elimination of all non-integer quantifiers in Set Theory statements.

I always wondered why math foundations as taken by logicians are so distant from the actual math practice. For instance, the cardinality theory - the heart of the Set Theory (ST) - is almost never used beyond figuring out which sets are countable and which are not.

I see the culprit in the blurred distinction between two types of collections different in nature. One are pure **classes** defined by math formulas. The other are **externals** that math handles but generally does not specify internally. Random sequences are an example.<sup>1</sup> Externals are proper sets and are the domain of ST variables. I postulate them to be hereditarily countable and independent of pure classes, i.e. have only finite Kolmogorov Information about them. **Math objects** (only informally called sets) are classes  $\{q: F_p(q)\}$  of sets  $q$  satisfying formulas  $F$  with external parameters  $p$ . Collections and properties of objects are treated as ones of those parameters. Excluded would be “Logics” objects of infinite “depth” such as, e.g., the collection of all true statements of Arithmetic.

**Cantor’s** axioms asserted that all Set Theory formulas define (quantifiable) sets. In effect, this allows formulas with quantifiers over formulas. This self-referential aspect turned out fatal.

Zermelo and Fraenkel reduced this aspect by (somewhat *ad-hoc*) restrictions on cardinality treatment by Cantor’s Axioms: Replacement must preserve it; only a separate Power Set increases it.

This cardinality focus has questionable relevance. Distinctions between uncountable cardinalities almost never looked at in math papers. Besides, generic subsets, from Power Set classes, with no other descriptions, find little use in math and greatly complicate its foundations.

And as all axiom systems have countable models, cardinalities feel like artifacts, designed to hide self-referential aspects. Many papers in Logic (e.g., [2, 3, 7]) aimed at isolating math segments where more ingenious proofs can replace the use of Power Set Axiom and its uncountable sets. But this breaks the unity of math: a very unfortunate effect. And complicating proofs is unattractive, too.

Expanding Set Theory with fancier formula types, axioms, etc. has no natural end. Benefits are little, and eventual consistency loss inevitable. Let us see how to drop any such excesses.

**Some Formalities.** In effect, I split math objects into a hierarchy of ranks, each quantifier binding only classes below one rank. Rank  $\leq k$  classes  $F_p^k$  are specified by a hereditarily countable (“proper” set) parameter  $p$  via the universal  $\Sigma_k^1$  formula  $F_p^k(q)$ . In  $p$ , a v. Neumann ordinal  $o(p)$  is included.  $F_p^k(q)$  defines membership  $F_q^k \in F_p^k$  conditioned on  $o(q) < o(p)$ . Membership is also extended by extensionality:  $F_{q'}^{k'} \in F_p^k$  if  $F_{q'}^{k'} = F_q^k$ , i.e. if membership relation graphs  $\boxed{\in(F)}$  of transitive closures of  $F_{q'}^{k'}$ ,  $F_q^k$  have a formula-defined isomorphism. Typical math concepts have straightforward translations. E.g., “ $x$  is in open  $P \subset \mathbb{R}$ ” can be a shorthand for “ $F_p^k(x)$ ”, where  $p$  specifies  $P \stackrel{\text{df}}{=} F_p^k$  as the set of its rational intervals” (Or  $P$  can be Borel, or whatever type clear from the context.)

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<sup>1</sup>In reality, external data end up finite. Yet, infinities are neat. Even for finite objects their termination points are often ambiguous and awkward to handle. As 0 is great simplifying approximation to negligibly small  $\epsilon$ , so is  $\infty$  for  $\frac{1}{\epsilon}$ . And note that the set  $\overline{\mathbb{R}}$  of infinitely long reals is compact, “less infinite” in that than  $\mathbb{Q}$ .

# 1 Purging the Tricksome Objects

Making explicit the internal to math (defined by formulas) and external (parameters based) aspects of math objects clarifies their nature, allows a better focus on issues brought by these different (albeit with many similarities) sources. “Pure” classes defined by parameter-free ST formulas  $F(n)$  are specific but tricky. Treated carelessly they easily bring paradoxes. They will not be quantified or put in unlimited definable collections: that would only extend the language of allowed formulas.

Proper sets are external data which math handles without specifying internally. Sets constitute the domain of ST variables. They can be put in collections based on properties and relations with other sets. This forms “mixed” classes, specified by ST formulas with some free variables taken for data parameters. Data may be chaotic, but “simpleminded,” playing no self-referential tricks.

General math objects – mixed classes – carry both tricks and chaos. But their tricks are limited to those of a single formula. Distinguishing sets from pure classes allows another radical insight: Even with infinite complexity, external sequences  $\alpha \in \Omega \stackrel{\text{df}}{=} \{0, 1\}^{\mathbb{N}}$  can have only finite dependence on pure sequences  $\Psi$ . Complexity theory (see sec. 2 for some background) allows to formalize that as having finite (small, really) mutual Kolmogorov Information  $\mathbf{I}(\alpha; \Psi) < \infty$ . Note, computable and random transformations can only add finite information about a given (e.g., by a formula) sequence.

[5] discusses the validity of this Independence Postulate (**IP**) for external data. (Besides, it is redundant for math objects  $F_p$  to duplicate formula-defined information in external parameters  $p$ .)

Mixed math objects would not encroach on infinite hierarchies. They reduce to a pair of sequences: one in  $\Delta_*^0 \stackrel{\text{df}}{=} \cup_n \Delta_n^0$  and another that carries no information about pure classes. This gives a way to handle infinitely complex sets, but reduce their quantifiers to those on **integers**. All with no seeming need to change anything in math papers, only reinterpret some formalities.<sup>2</sup>

By **IP**, sequences can double as both pure classes and sets only if computable. So, **IP** conflicts with Replacement axioms (RA) for sets. (For classes, RA are merely definitions.) Thus I restrict RA axioms to just one: Classes  $s$  with  $\in(s)$  **computably enumerable (c.e.)** from a set are sets. Instead the Foundation axiom becomes a family (like Induction axioms): one for each class of ordinals. And “Primal Chaos” axiom (**PX**) is added: “ $\in(s)$  for each set  $s$  is enumerable from even-indexed digits of some  $\beta$  that is Martin-Lof random:  $\beta \in \mathbf{R}^\lambda \subset \Omega$ .” (For classes, i.e. in ZFC, it is a theorem: see [6].)

Remarkably, **IP** opens a way to eliminate all non-integer quantifiers, It excludes  $\alpha$  satisfying  $F(\alpha, \beta)$  unless such  $\alpha$  reduce to a positive fraction of those  $\gamma \in \Omega$  from which  $\beta$  is computable, too. The converse requires more axioms, like **PX**. Let us call IPH the system being constructed.

Its consistency seen from its ZFC model. As models can always be made countable, we can have it in “internal style” (of IPH plus a model of Peano Arithmetic). It would list the IPH sets using a  $\gamma \in \mathbf{R}^\lambda$  that itself respects **IP/PX**, i.e. is **generic**: outside all classes  $X \in \Delta_*^0$  of measure  $\lambda(X) = 0$ . Model’s sets are those enumerable from  $\gamma^{(k)} : i \mapsto (i \bmod k)\gamma_i$  for some  $k$ . Such models respect a c.e. family IMA of “Internal Models axioms”:  $F \Leftrightarrow \lambda(\{\gamma : F'(\gamma)\}) > 0$  where  $F'$  has all 2nd order variables  $\alpha_i$  in  $F$  replaced by  $A_i(\gamma^{(k_i)})$ , with  $k_i$  and algorithms  $A_i$  quantified as integers. Cf.[4].<sup>3</sup>

<sup>2</sup>Some statements can be a meta-theorem: a family with formula parameter  $F$ , as done now by Category Theorists.

<sup>3</sup>And IMA may be just theorems. Any consistent with IPH statement  $G$  holds in a countable model  $M$  of IPH axioms. Let such  $G$  be  $F \& \lambda(\{\gamma : F'(\gamma)\}) = 0$ . Then  $M$  respects  $G$ , **PX**, and **IP** for for some  $\Psi \in \Delta_*^0 \cap \mathbf{R}^\lambda$ , such that  $\forall \gamma ((G \& F'(\gamma)) \Rightarrow (\gamma, \Psi) \notin \mathbf{R}^\lambda)$ . For a contradiction we need to express  $M$  in an “internal style” with its sets being those computable from  $\gamma^{(k)}$  where  $(\gamma, \Psi) \in \mathbf{R}^\lambda$ . For some fixed  $c$ , let  $\mathbf{R}_\Psi^\lambda \stackrel{\text{df}}{=} \{\alpha : (\Psi, \alpha) \in \mathbf{R}_c^\lambda\}$ .  $M$  lists all sets; combine the first  $i$  of them into  $\alpha_i \in \mathbf{R}_\Psi^\lambda$ , so that  $\mathbf{K}(\alpha_i | \alpha_{i+1}) < \infty$ . We need to merge all  $\alpha_i$  into (so reduce to) a single  $\alpha \in \mathbf{R}_\Psi^\lambda$ . This takes constructing a sequence of  $\beta_i = u(i, \beta_{i+1}) \in \mathbf{R}_\Psi^\lambda$ , each Turing-equivalent to  $\alpha_i$ .

## 2 Kolmogorov Information: A Brief Background

I freely use concepts involving quantifiers only over hereditarily finite sets. I presume reader knows how to develop such basics as, e.g., algorithms, and do not try to work out elegant ways to define them directly in membership terms. Below  $\|x\| \stackrel{\text{df}}{=} n$  for  $x \in \{0, 1\}^n$  and  $\|t\| \stackrel{\text{df}}{=} \lceil \log_2 t \rceil - 1$  for  $t \in \mathbb{R}^+$ .

**Uniform measure** on  $\Omega = \{0, 1\}^{\mathbb{N}}$  (or on  $\Omega^k \simeq \Omega$ ) is  $\lambda(x \Omega) \stackrel{\text{df}}{=} 2^{-\|x\|}$ .

**Partial continuous transforms (PCT)** on  $\Omega$  may fail to narrow-down the output to a single sequence, leaving a compact set of eligible results. So, their graphs are compact sets  $A \subset \Omega \times \Omega$  with  $A(\alpha) \stackrel{\text{df}}{=} \{\beta : (\alpha, \beta) \in A\} \neq \emptyset$ . Singleton outputs  $\{\beta\}$  are interpreted as  $\beta \in \Omega$ .

**Preimages**  $A^{-1}(s) \stackrel{\text{df}}{=} \{\alpha : A(\alpha) \subset s\}$  of all open sets  $s \subset \Omega$  in any PCT  $A$  are open.

**Closed**  $A$  also have closed preimages of all closed  $s$ . For  $x \in \{0, 1\}^*$  let  $x^A \stackrel{\text{df}}{=} \{\alpha : A(\alpha) \in x\Omega\}$ .

**Computable PCTs** have algorithms enumerating the clopen subsets of  $\Omega^2 \setminus A$ .  $U(p\alpha)$  is a universal PCT. It computes  $n = \|x\|$  bits of  $A_p(\alpha) \subset x\Omega$  in  $t_{p\alpha}(n)$  steps (and  $U(p) \in \mathbb{N}$  in  $t_p$  steps).

A **computably enumerable** (c.e.) function to  $\mathbb{R}^+$  is sup of a c.e. set of basic continuous ones.

**Dominant** in a Banach space  $C$  of functions is its c.e.  $f \in C$  if all c.e.  $g$  in  $C$  are  $O(f)$ .

Such is  $\sum_i g_i / (i^2 + i)$  if  $(g_i)$  is a c.e. family of all c.e. functions in the unit ball of  $C$ .

**$\lambda$ -test** is  $\mathbf{d}(\alpha) \stackrel{\text{df}}{=} \|\mathbf{T}(\alpha)\|$  for a c.e.  $\mathbf{T} : \Omega \rightarrow \mathbb{R}^+$ ,  $\lambda(\mathbf{T}) \leq 1$ , dominant in  $\mathbf{L}^1(\Omega, \lambda)$ .

**Martin-Lof  $\lambda$ -random** are  $\alpha \in \mathbf{R}^\lambda \stackrel{\text{df}}{=} \mathbf{R}_\infty^\lambda$ , where  $\mathbf{R}_c^\lambda \stackrel{\text{df}}{=} \{\alpha : \mathbf{d}(\alpha) < c\}$  (compact for  $c \in \mathbb{N}$ ).

$x \sqsubset \alpha$  means  $\alpha \in x\Omega$ . Let  $\mathbf{M}(x) \stackrel{\text{df}}{=} \lambda(U^{-1}(x\Omega))$ .  $\mathbf{R}^\lambda$  consists of all  $\gamma$  with  $\sup_{x \sqsubset \gamma} \frac{\mathbf{M}(x)}{\lambda(x\Omega)} < \infty$ .

**Mutual Information** (dependence)  $\mathbf{I}(\alpha_1 : \alpha_2)$  is  $\min_{\beta_1, \beta_2} \{\mathbf{d}(\beta_1, \beta_2) : U(\beta_i) = \alpha_i\}$ .

**Measuring classes.** For a sequence of clopen  $F_i \subset \Omega$  with  $\mathbf{M}(F_i \oplus F_{i+j}) < 2^{-i}$ , let open  $F_i^+ \stackrel{\text{df}}{=} \bigcup_{j > i} F_j$ , compact  $F_i^- \stackrel{\text{df}}{=} \bigcap_{j > i} F_j$ ;  $F^- \stackrel{\text{df}}{=} \bigcup_i F_i^- \subset F^+ \stackrel{\text{df}}{=} \bigcap_i F_i^+$ . Any  $F \in \Delta_*^0$  has such  $(F_i) \in \Delta_*^0$  with  $F^- \subset F \subset F^+$ . Note that  $F^+ \setminus F^-$  contains no *generic*  $\gamma$ . If  $F \subset \mathbf{R}^\lambda$ ,  $\lambda(F) = 0$  then also  $\mathbf{M}(F_i) < 2^{-i}$ , implying  $\mathbf{I}(\gamma : (F_i)_i) = \infty$  for all  $\gamma \in F$ . Thus, any arithmetic class  $F$  includes no generic  $\gamma \in \mathbf{R}^\lambda$ , if  $\lambda(F) = 0$ , and includes all such  $\gamma$  or none if  $F$  is invariant under single digit flips.

### 2.1 Weak Truth-table (Closed PCT) Reductions to Generic Sequences

Let  $s_t^n \stackrel{\text{df}}{=} \lambda(\{\alpha : t_\alpha(n) < t\})$ ,  $s_\infty \stackrel{\text{df}}{=} \inf_n s_\infty^n$ . The function  $\tau_r^n \stackrel{\text{df}}{=} \min\{t : s_t^n > r\}$ ,  $r \in \mathbb{Q}$ , is computable.

Let  $\mathbf{I}(\alpha : s_\infty) < \infty$ . Then  $s_\alpha \stackrel{\text{df}}{=} \liminf_n s_{t_\alpha(n)}^n < s_\infty$  as  $s_\infty \notin \Delta_2^0$ . Take  $r \in (s_\alpha, s_\infty)$  and a monotone infinite sequence  $n_i^\alpha$  of all  $n$  with  $t_\alpha(n) < \tau_r^n$ . Then  $\forall^\infty i \min\{p : t_p > n_i^\alpha\} < i\|i\|^2$ .

$U_c(\alpha) \in \{\#, 0, 1\}^{\mathbb{N}}$  avoids divergence by diluting  $U(\alpha)$  with  $\min_{p < i\|i\|^2 + c} \{t_p : t_p > n_{i+1}^\alpha\} \leq \infty$  blanks  $\#$  after  $n_i^\alpha$ -th bits.  $U_c$  carries no extra information of  $\alpha$  absent in  $U(\alpha)$ ,  $n^\alpha$ . As  $U_c$  never diverges,  $\mu \stackrel{\text{df}}{=} U_c(\lambda)$  is computable and  $U'(\alpha) \stackrel{\text{df}}{=} \mu([\#^{\mathbb{N}}, U_c(\alpha)])$  maps  $\mathbf{R}^\lambda$  to  $\mathbf{R}^\lambda \cup U(\mathbb{N})$ .

From  $\beta = U'(\alpha) \in \mathbf{R}^\lambda$  we recover  $U_c(\alpha)$  and  $u(\beta) \stackrel{\text{df}}{=} U(\alpha)$ . PCT  $u$  is closed (a w.t.t. reduction) as the input segments it uses are only slightly longer (by codes  $p$  for  $n^\alpha$  bounds) than the output's.

(Viewing  $\#^+ \{0, 1\}^+$  segments of  $U_c(\alpha)$  as integers makes  $\mu$  a computable (on  $\mathbb{N}^*$  prefixes) distribution on (finite and infinite) sequences of integers.  $U'$  gives them short codes.)

### 2.2 To Eliminate 2nd Order Quantifiers with IP

**Simplifications with PX.** To *reduce* all ST predicates or classes, i.e., express them using only integer quantifiers, suffices to reduce  $\{\beta : \exists \alpha (\alpha, \beta) \in P\}$ , for any reduced  $P$ , where  $\beta \in \Omega^k \simeq \Omega$ . Note,  $\exists \alpha \in \Omega P(\alpha) \Leftrightarrow \exists \alpha \in \mathbf{R}_0^\lambda P(u(\alpha))$ , so  $\alpha, \beta, \gamma$  below will range only over  $\mathbf{R}^\lambda$ . Also, any pair  $(\alpha, \beta)$  is  $(A(\gamma), u(\gamma))$ , so we need to reduce  $u(P')$  for reduced  $P' \stackrel{\text{df}}{=} \{\gamma : \exists \alpha (A(\gamma), u(\gamma)) \in P\}$ .

Let  $\lambda_x(Q) \stackrel{\text{df}}{=} \frac{\lambda(Q \cap x^u)}{\lambda(x^u)}$ ,  $\lambda_\beta(Q) \stackrel{\text{df}}{=} \liminf_{x \sqsubset \beta} \lambda_x(Q)$  be  $\lambda$  on  $\gamma$  conditioned on  $u(\gamma) = \beta$ . Note:  $\lambda_\beta(Q) > t$  is reduced if  $Q$  is. **IP** assures  $\lambda_\beta(P') > 0$  for  $\beta \in u(P')$ , so taking  $P_k \stackrel{\text{df}}{=} (P' \cup \{\gamma : \lambda_{u(\gamma)}(P') \leq 2^{-k}\})^-$  turns  $\beta \in u(P')$  into  $\exists k (\beta \in u(P_k) \& \lambda_\beta(P') > 2^{-k})$ . So we are to reduce  $u(P_k)$  or  $F \stackrel{\text{df}}{=} u(P_k) \cap \mathbf{R}^\lambda$ . Note,  $\forall x \lambda_x(P_k) > 2^{-k}$ , so it seems  $F = \mathbf{R}^\lambda$ . But proving that is complicated by one-way functions issue:

**Computably one-way** functions on  $S$ , are closed computable PCT  $A$ , s.t.  $\lambda(x^A \cap S) = O(\lambda(x\Omega))$ ,  $\lambda(S) > 0$ , and no computable  $B$  invert  $A$ , i.e.  $\lambda^2(\{(\beta, \gamma): \alpha \stackrel{\text{df}}{=} B(\beta, \gamma) \in S, A(\alpha) = \beta\}) = 0$ .

Had no such OWF existed (though they do: [1]), **PX** would give a reverse of the above exclusion by **IP**, thus eliminate all non-integer quantifiers. But handling OWFs demands more tools, such as, e.g., IMA of Internal models. Yet, adding such a family as axioms strikes me as less elegant or intuitive than a single neat axiom, such as **PX**, would be. Much is unclear to me yet about this.

### 3 Some Discussion

Cantor Axioms license on formula-defined sets led to fatal consistency problems. Zermelo-Fraenkel's (somewhat *ad-hoc*) cardinality-based restrictions diffused those but left intact their self-referential root. The result was a Babel Tower of cardinalities, other hierarchies finding little relevance in math. A clean way out may be recognizing the distinction between internal to math collections specified by its formulas and external ones that math handles as values of variables, without specifying.

Internal collections have a limited hierarchy: the type of allowed formulas is clear-cut. Any extension would make a new theory, with its own clear limits. External objects would be fully independent of internal ones: having finite information about them. Complexity theory allows to formalize that, justify the validity for "external data," and use that for simplifying math foundations.

Generic math objects are collections specified by formulas with external sets as parameters. Collections of them are treated as collections of those parameters. So, any such object would rely on a single formula, not on all of them, thus excluding back-door extensions of formula language.

The uniform set concept of all math objects with no explicit types hierarchy is luring but illusive. The hierarchy of ever more powerful axioms, models, cardinals remains, if swiped under the rug. Making it explicit and matching the math relevance may be a path to simpler foundations.

What is left out? – "Logical" sets, related to infinite hierarchies of formulas, such as "the set of all true sentences of Arithmetic." Those should be subject of math foundations. Theories cannot include their own foundations. Math Logic then could focus on math rather than on itself.

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### References

- [1] George Barmpalias, Xiaoyan Zhang. 2024. Computable one-way functions on the reals. <https://arxiv.org/abs/2406.15817> . Also: Peter Gacs. 2024. Personal communication.
- [2] H.Friedman. 1974. Some systems of second-order arithmetic and their use. *ICM proc.* **1**:235-242.
- [3] Randall Holmes. 2006. Alternative Set Theories. *Stanford Encyclopedia of Philosophy*. <https://plato.stanford.edu/entries/settheory-alternative/>
- [4] Leonid A. Levin. 1976. On the Principle of Conservation of Information in Intuitionistic Mathematics. *Soviet Math. Dokl.* **17**/2:601-605 = *DAN SSSR* **227**/6:1293-1296.
- [5] Leonid A. Levin. 2013. Forbidden Information. *JACM*, **60**/2. Sec. "The Independence Postulate" <https://arxiv.org/abs/cs/0203029>
- [6] Leonid A. Levin. 2022. Gacs-Kucera Theorem. *TCS*, **929**:172-173. <https://arxiv.org/abs/2207.12489>
- [7] Stephen G. Simpson. 2009. *Subsystems of Second Order Arithmetic*. Cambridge U. Press.
- [8] E.Zermelo. 1908. Untersuchungen über die Grundlagen der Mengenlehre. *Math. Ann.* **65**:261-281.

## A Appendix. Some Other IP Applications

**Foundations of Probability.** Paradoxes in its application led to the K-ML randomness concept  $\mathbf{R}^\lambda$ . **IP** clarifies its use: For any  $S \subset \Omega$ :  $\lambda(S)=0$  if and only if  $\exists \sigma \ S \cap \mathbf{R}^\lambda \subset \{\gamma : \mathbf{I}(\gamma : \sigma) = \infty\}$ .

**Goedel Theorem Loophole.** Goedel writes:

“It is not at all excluded by the negative results mentioned earlier that nevertheless every clearly posed mathematical yes-or-no question is solvable in this way. For it is just this becoming evident of more and more new axioms on the basis of the meaning of the primitive notions that a machine cannot imitate.”

**No way !** Let a predicate  $P$  on  $\{0,1\}^n$  extend “proven/refuted” partial predicate of Peano Arithmetic. Let  $r_n$  be the  $n$ -bit prefix of a c.e. real  $r = \min \mathbf{R}_0^\lambda$ . Then  $\mathbf{I}(P : r_n) = n \pm O(\log n)$ .

## B Appendix. ZFC Axioms

ZFC axioms are sometimes given to undergraduates in an unintuitive, hard to remember list. Setting them in three pairs seems to help intuition.

Sets with a given ST property  $F$  (possibly with parameters  $c$ ) are said to form a **class**  $\{x : F_c(x)\}$ . They may or may not form a set, but only sets are the domain of ZFC variables.

1. **Membership chains: sources, sinks.** (1b anti-dual to 1a):

1a. **Infinity** (a set with no membership source):

$$\boxed{\exists S \neq \emptyset \ \forall x \in S \exists y \in S (x \in y)}$$

1b. **Foundation** (each set has sinks: members disjoint with it):

$$\boxed{\neg \exists S \neq \emptyset \ \forall x \in S \exists y \in S (y \in x)}$$

2. **Sets with formula-defined membership:**

2a. **Extensionality:** (content identifies sets uniquely):

$$\boxed{x \supset y \supset x \in t \Rightarrow y \in t}$$

2b. **Replacement:**<sup>4</sup>

$$\boxed{(\forall x \exists Y \supset R_c(\{x\})) \Rightarrow \forall X \exists Y \supset R_c(X) \supset Y}$$

3. **Functions Inverses.**  $f^{-1} \stackrel{\text{df}}{=} \{g : f(g(x))=x, \text{Dom}(g)=\text{Im}(f)\}$ :

3a. **Powerset** ( $f^{-1} \subset G$  is a set, as  $g \subset h = f^T$ ):

$$\boxed{\forall h \exists G \forall g (g \subset h \Rightarrow g \in G)}$$

3b. **Choice**<sup>5</sup> ( $f^{-1}$  is not empty):

$$\boxed{\forall f \exists g \in f^{-1}}$$

**The modifications discussed above include:**

1. Restrict Replacement to computable  $R$ , drop Power Set;
2. add **IP**, **PX**, strengthened;
3. extend Foundation to classes of sets (as a family of axioms, like Peano Induction);
4. some (unclear yet) replacement for Choice.<sup>6</sup>  
(May be dropping it, or adding to the language a postulated (not described) class mapping countable v. Neumann ordinals onto reals, implying continuum hypothesis, too.)

Math objects are classes  $\{q : F_p(q)\}$  of sets  $q$  satisfying formulas  $F$  with external parameters  $p$ . Collections of objects are treated as collections of those parameters, with conditions for Foundation and Extensionality. Quantifiers bind parameters, not properties  $F$ .

<sup>4</sup>An axiom for each ST-defined relation  $R_c(x, y)$ .  $R_c(X) \stackrel{\text{df}}{=} \{y : \exists x \in X R_c(x, y)\}$ .

<sup>5</sup>The feasibility of computing inverses is the most dramatic open Computer Theory problem.

<sup>6</sup>I am not clear yet on how to handle some math theorems that depend on Axiom of Choice, as it provides classes not specified by ZF formulas. One option may be extending the language with a postulated (not otherwise described) class that maps countable v. Neumann ordinals onto reals (implying Continuum Hypothesis, too). But any such options would require careful analysis.