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SOME SYNTACTIC THEOREMS ON THE CALCULUS OF FINITE PROBLEMS OF Ju. T. MEDVEDEV

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In the well-known work of A. N. Kolmogorov [1] one can find an interpretation of intuitionistic logic [2] as a calculus of problems. This idea was subsequently made more precise from various points of view (cf. for example [3-5]). Another refinement of this interpretation of A. N. Kolmogorov was indicated in [6] and extended later in [7,8]. The present work presents some results about the logic of finite problems constructed in [6]. Familiarity with the concepts and terminology of the article [6] will be assumed. We will show (in Theorem 2) that the logic of finite problems is complete in a well-defined sense. Moreover, a syntactic description of this logic will be given (Theorem 4) and a calculus will be constructed (containing, it is true, a certain additional nonfinite rule of inference D) whose class of deducible formulas coincides with the class of finitely valid formulas (Corollary 1).

We first give five definitions.

Definition 1. By a *logic* we will understand any class of formulas of the propositional calculus that contains all the axioms of the intuitionistic propositional calculus and which is closed under the application of modus ponens and substitution.

Definition 2. By an *α-logic* we understand any logic that contains the formula $\alpha = (\neg x \supset (y \vee z)) \supset ((\neg x \supset y) \vee (\neg x \supset z))$.

Definition 3. A *weakly constructive logic* is a logic that does not contain any formula that is the disjunction of formulas that are not classically deducible.

Definition 4. We define the *rank* of a formula of the propositional calculus inductively as follows: a) any formula of the form $\neg\phi$ has rank 1; b) if ψ has rank 1 and ϕ is an arbitrary formula, then the formula $\phi \supset \psi$ has rank 1; c) if ϕ is a formula of rank k and ψ is a formula of rank l, then $\phi \vee \psi$ is a formula of rank $k + l$, $\phi \,\&\, \psi$ is a formula of rank $k \cdot l$, and $\phi \supset \psi$ is a formula of rank l^k; d) all other formulas have infinite rank.

Definition 5. A logic is called *finite* if for every formula $\phi(x_1, x_2, \cdots, x_n)$ that does not belong to it there exist formulas $\psi_1, \cdots, \psi_n$ of finite rank such that the formula $\phi(\psi_1, \cdots, \psi_n)$ also does not belong to the logic.

Theorem 1. *The class of finitely valid formulas* (*i.e. the formulas that are always decidable in the sense of* [6]) *is a weakly constructive finite α-logic.*

Theorem 2. *The class of finitely valid formulas is the largest weakly constructive α-logic.*

Theorem 3. *The class of finitely valid formulas is the smallest finite α-logic.*

Corollary 1. *Let U be the calculus containing the axioms and rules of inference of the intuitionistic propositional calculus and, in addition, the formula α as axioms and the following rule of inference:*

$$\overline{V} = (x_n x_n^0)^{-\mu} R^{-2\gamma} \sum_{k=0}^{\infty} \frac{\rho^k}{(1-\gamma)_k k!} H_2\left(\gamma - k, \beta, \mu, 1-\mu, a; \frac{1}{\omega}, -\lambda\right),$$

$$R^2 = \sum_{i=1}^{n} (x_i - x_i^0)^2 - (s - s_0)^2, \ 4\rho = b^2 R^2, \ 4ss_0\omega = R^2, \ 4x_n x_n^0 \lambda = -R^2,$$

where F_3 and H_2 are the hypergeometric functions of Appell and Horn.

For the study of properties of the confluent hypergeometric functions of three variables obtained in this way, we can make use of their integral transformations

$$U = \delta (ss_0)^{-\beta} (x_n x_n^0)^{-\mu} R^{2N-2} \int_0^1 [\xi(1-\xi\omega)]^{-\beta} (1-\xi)^{N+\beta-2} Q_0(\xi) d\xi, \tag{23}$$

$$\overline{V} = \delta_1 (x_n x_n^0)^{-\mu} R^{-2\gamma} \int_0^1 [\xi(1-\xi)]^{\beta-1} \left(1 - \frac{\xi}{\omega}\right)^{-\gamma} \overline{Q}_1(\xi) d\xi, \quad (\beta > 0),$$

$$Q_0(\xi) = \Xi_2[\mu, 1-\mu, N+\beta-1; \lambda(1-\xi), \rho(1-\xi)],$$

$$\overline{Q}_1(\xi) = \Xi_2\left[\mu, 1-\mu, 1-\gamma; \lambda\left(1-\frac{\xi}{\omega}\right), \rho\left(1-\frac{\xi}{\omega}\right)\right],$$

$$\Gamma(1-\beta)\Gamma(N+\beta-1)\delta = \Gamma(N), \ \Gamma^2(\beta)\delta_1 = \Gamma(a),$$

with the condition that λ and ω do not fall outside the limits of the domain of convergence of the Humbert series for Ξ_2, and in (23) $1 - N < \beta < 1$. In particular, when $\mu = 0$, U and $\overline{V}$ take the form

$$U = (ss_0)^{-\beta} R^{2N-2} \Xi_2(\beta, 1-\beta, N; \lambda, \rho),$$

$$\overline{V} = R^{-2\gamma} H_3\left(\gamma, \beta, a; \frac{1}{\omega}, -\rho\right).$$

In the same way (4a), (6b), (15) and (16) of [6] are generalized.

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D. *If* $\phi(x_1, \cdots, x_n)$ *is a formula such that for any formulas* $\psi_1, \cdots, \psi_n$ *of finite rank one has* $\vdash \phi(\psi_1, \cdots, \psi_n)$, *then* $\vdash \phi(x_1, \cdots, x_n)$.

The class of deducible formulas of the calculus U coincides with the class of finitely valid formulas.

Theorem 4. *The class of finitely valid formulas is the only weakly constructive finite* α*-logic.*

This theorem follows trivially from Theorems 2 and 3.

We now formulate six lemmas needed in the proofs of Theorems 2 and 3. Each of these lemmas follows easily from the previous one.

Lemma 1. *For any formula* ϕ *of finite rank there exists a formula* ψ *of the form* $\psi = \neg\psi_1 \vee \neg\psi_2 \cdots \vee \neg\psi_k$ *such that the formula* $(\phi \sim \psi)$ *belongs to every* α*-logic.*

The proof is by induction on the rank of the formula.

Lemma 2. *Every logic contains all the classically derivable formulas of the form* $\neg\psi$.

This lemma is a direct corollary of the well-known theorem of V. I. Glivenko [9] about the intuitionistic derivability of any classically derivable formula that has the form of a negation.

Lemma 3. *Every logic contains a formula of the form* $\neg\psi_1 \vee \cdots \vee \neg\psi_k$, *if for some* i, $1 \le i \le k$, *the formula* $\neg\psi_i$ *is classically deducible.*

Lemma 4. *Every weakly constructive logic contains the formula* $\neg\psi_1 \vee \cdots \vee \neg\psi_k$ *if and only if for some* i, $1 \le i \le k$, *the formula* $\neg\psi_i$ *is classically deducible.*

Lemma 5. *Every* α*-logic contains all the finitely valid formulas of finite rank.*

This follows from Lemmas 1 and 3.

Lemma 6. *Every weakly constructive* α*-logic contains a formula* ψ *of finite rank if and only if* ψ *is finitely valid.*

This follows from Lemmas 1 and 4.

We now prove Theorems 2 and 3 with the help of these lemmas.

Proof of Theorem 2. Let $\phi(x_1, \cdots, x_n)$ be refutable. Then there exist formulas $\psi_1, \cdots, \psi_n$ of finite rank such that the formula $\phi(\psi_1, \cdots, \psi_n)$ is refutable (by Definition 5 and Theorem 1). But then by Lemma 6 the formula $\phi(\psi_1, \cdots, \psi_n)$ does not belong to any weakly constructive α-logic. This means that the formula $\phi(x_1, \cdots, x_n)$ also possesses this property, which is what we desired to show.

Proof of Theorem 3. Assume that $\phi(x_1, \cdots, x_n)$ does not belong to any finite α-logic. Then there exist formulas $\psi_1, \cdots, \psi_n$ of finite rank such that the formula $\phi(\psi_1, \cdots, \psi_n)$ (of finite rank) also does not belong to the logic. Then $\phi(\psi_1, \cdots, \psi_n)$ is refutable (by Lemma 5). But then $\phi(x_1, \cdots, x_n)$ is also refutable, which is what was required.

Results close to those just presented have also been obtained independently by V. A. Jankov.

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