Set Theory in the Foundation of Math; Internal Classes and External Sets

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More details in:
www.cs.bu.edu/fac/Lnd/expo/sets/
or (might be older) in:
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Set Theory in Math Foundations; Internal Classes, External Sets.

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Set Theory: Some History, Self-Referentials

Cantor's Axioms: All Set Theory formulas define sets. In effect, formulas with quantifiers over formulas: A fatal **self-referential** aspect.
Zermelo, Fraenkel restricted cardinality in Cantor's Axioms: **Replacement** preserves it. Only a separate **Power Set** increases it.

Somewhat *ad-hoc* as math foundations. And cardinality focus has questionable relevance. Distinctions between uncountable cardinalities are almost never looked at in math papers.

Usual math sets have special types: countable, compact, open, occasionally Borel, rarely projective, etc. Generic subclasses from Power Set with no other descriptions, find little use in math, and greatly complicate its foundations.

All consistent axiom systems have countable models. Cardinalities look like an artifact, designed to hide some self-referential aspects.

To Handle Concerns; Cardinalities

Expanding Set Theory with more types of formulas, axioms, etc. has no natural end. Few benefits, eventual consistency loss inevitable. ZF-restricted self-referentials, such as implicit quantifiers over formulas, brought so far no inconsistencies, but find little math use either.

Logicians: Isolate math segments where more ingenious proofs can replace the use of Power Set Axiom and its uncountable sets.

Math folk: Bad to mess with math unity. Keep whole its monumental structure ! And better not to complicate proofs.

Computer **T**(errorist): Timidity never works ! Reject infinite sets.

Dear Comp. **T**heorist: Agreed on timidity, but drop your errorist aspect ! Infinities are neat: Approximating ε with 0 and $\frac{1}{\varepsilon}$ with ∞ is a great simplification. And handling (often ambiguous) termination points of objects is awkward. And $\overline{\mathbb{R}}$ is compact, "less infinite" in that than $\overline{\mathbb{Q}}$.

Going at the Self-Referential Root

To avoid self-referentials: separate the domains of variables in math properties from properties themselves in the objects they define.

Externals: sets handled as values of variables, rather than as internally math-specified ones, e.g., random sequences. Mark them apart from **classes**: collections defined by math formulas.

Math objects (only informally called sets) are classes of sets specified by formulas with external parameters. Collections of objects are treated as collections of those parameters. Quantifiers bind parameters, not formulas.

This requires almost no changes in actual math papers: only reinterpreting some formalities.

Radical Computer Theorist Hits Back

Independence Postulate

Even with infinite complexities,
external objects have finite (small, really)
information about formula-defined classes.

Besides, it is redundant for math objects
to duplicate in the external parameters
their formula-defined information.

Complexity theory allows to formalize that,
justify the validity for "external data", and
use that for simplifying math foundations.

Gives a way to handle infinitely complex sets,
but reduce all quantifiers to bind only **integers**.
All without restricting anything used in math,
except for some esoteric or foundational issues.

Some Complexity Background

Distributions: p:{0,1}$^*\to\mathbb{R}^+$: enumerable from below, and summable ($\sum_x p(x) \le 1$) functions.

Dominant Distribution $\mathbf{m}$: $p{=}O(\mathbf{m})$ for all p.

Complexity: its entropy: $\mathbf{K}(x) \stackrel{\text{df}}{=} -\lceil \log \mathbf{m}(x) \rceil$.
(It equals the shortest length of prefixless programs generating x.)

Kolmogorov–Martin-Lof Randomness:

Flat on $\mathbf{\Omega} \stackrel{\text{df}}{=} \{0,1\}^{\mathbb{N}}$ measure: $\boldsymbol{\lambda}(x\mathbf{\Omega}) \stackrel{\text{df}}{=} 2^{-\|x\|}$.

$\boldsymbol{\lambda}$-Rarity: $\mathbf{d}(\alpha) \stackrel{\text{df}}{=} \sup_n \{n - \mathbf{K}(\alpha_{[n]})\}$.
Randomness: $\mathbf{R}_c \stackrel{\text{df}}{=} \{\alpha : \mathbf{d}(\alpha) < c\}$, $\mathbf{R} \stackrel{\text{df}}{=} \mathbf{R}_\infty$.

Mutual **Information:**
$\mathbf{I}(\alpha_1 : \alpha_2) \stackrel{\text{df}}{=} \min_{\beta_1,\beta_2}\{\mathbf{d}((\beta_1,\beta_2)) : \alpha_i = u(\beta_i)\}$.

Independence Postulate

IP: $\boxed{\forall\alpha\, \mathbf{I}(\alpha{:}\, F) < \infty}$ (A family of axioms, one for each property $F \in \Delta^0_* \stackrel{\text{df}}{=} \cup_n \Delta^0_n \subset \mathbf{\Omega}$.)

(By **IP**, pure classes $\alpha{\in}\Delta^0_*$ double as sets only if computable: a sort of Church's Thesis.)

Justifications and Applications

Conservation Theorems: no processing of α, algorithmic, or random, or mixed, increases $\mathbf{I}(\alpha{:}F){+}O(1)$. Arguably, no real process can.

No loss of expressive power: formulas can by themselves handle information from other formulas, no need to duplicate in parameters.

At the end, I will mention some other, not ST, powerful applications of **IP**.

The Formalities

Many sets, such as linear spaces, manifolds, etc. would then be interpreted as classes, specified by formulas with parameters.

The formula defines membership relation on the transitive closure of the class as a relation between parameters specifying its members.

The parameters are sets (externals): hereditary countable, respecting Independence Postulate. Quantifiers bind parameters, **NOT** formulas.

Papers may have families of theorems with formula parameters as used now for Categories or for families of axioms, like Induction Axioms. Those may be meta-mathematical statements. Or, variables for formulas can be allowed, as unquantifiable, except for the implicit universal quantifiers over all free variables in a sentence.

Reducing All Quantifiers to those on Integers

IP opens a way: excludes $\alpha \in F \in \Delta^0_*$ unless such α reduce to a positive fraction of all sequences. (Note: $\boldsymbol{\lambda}(F) > t$ has only integer quantifiers.)

But what about the reverse?

Pχ (**Primal Chaos**) axiom (= Gacs-Kucera theorem in ZF): "Each $\alpha \in \boldsymbol{\Omega}$ reduces to even-indexed digits of some random $\beta \in \mathbf{R}$."

Note: a random sequence respects **IP** if and only if it is **generic**, i.e., is outside of all arithmetic classes of measure 0.

(This may make **IP** more intuitive for random sequences. And all sequences reduce to them.)

Consistency:
Models, Countable, Internal.

Our axioms are consistent, having, as all consistent theories do, countable models.

Each model is **directed** under reducibility. That is, with any two sequences it includes another they are both computable from, and also all sequences computable from them.

A countable model has a **reduction basis**: a chain $\gamma^{(k)}$, each computable from $\gamma^{(k+1)}$, and all model's sequences computed from them. By **IP**, **P**$\boldsymbol{\chi}$, they all reduce to **generic** sequences (and can be taken Turing-equivalent to them).

We can view $\gamma^{(k)}$ as parts of some combined γ, with dropped γ_i for, say, i divisible by 2^k. Call such models **internal** if γ is generic itself.

Internal models respect a family **IM** of axioms: $F \Leftrightarrow \boldsymbol{\lambda}(\{\gamma\colon F_\gamma\}) > 0$, where F_γ has all variables $\alpha_i \in \boldsymbol{\Omega}$ in the sentence F replaced by $A_i(\gamma^{(k_i)})$; k_i and algorithms A_i treated as integers. This eliminates all non-integer quantifiers in all F !

A Problem: One-Way Functions

Adding to **IP** such a whole family of axioms does not strike me as really elegant, intuitive. I hoped, adding a single Gacs-Kucera Theorem as a Set Theory axiom would suffice. By **IP**, $(\exists\alpha P(\alpha,\overline{\beta}) \Rightarrow \boldsymbol{\lambda}(\{\gamma\colon P(u(\gamma))\} \mid u(\gamma){\in}\,\mathbf{\Omega}{\times}\overline{\beta}) > 0$. But deriving "$\Leftarrow$" via $\mathbf{P}\boldsymbol{\chi}$ meets an obstacle:

Recursively One-Way functions (discovered by G. Barmpalias, P. Gacs, X. Zhang in 2024). Let f preserve $\boldsymbol{\lambda}$, i.e. $\boldsymbol{\lambda}(f^{-1}(x\,\mathbf{\Omega})) = O(\boldsymbol{\lambda}(x\,\mathbf{\Omega}))$. It is **one-way** if no g inverts it (f, g assumed computable): $\boldsymbol{\lambda}^2(\{(\beta,\gamma) : f(g(\beta,\gamma)){=}\beta\}) = 0$.

Handling OWFs demands more tools.
A single axiom would be more elegant
and intuitive than the whole **IM** family.
I have some ideas but the problem is still open.

Takeaway: the Issues

1. Cardinality-based ZF restrictions of Cantor's Axioms defuse fatal problems but retain their self-referential source. A bit *ad-hoc*. Results in Babel Towers of cardinalities, other hierarchies that find little relevance in math. Just a case of uncoiling a vicious circle into a vicious spiral.

2. A number of papers replace, in segments of math, the Power Set with more elaborate proofs. But this breaks the unity of math, so does not seem to be the right solution.

3. I blame the blurred distinction between the internal (math-defined) and the external (the domain of variables) aspects of math objects.

4. Extending Set Theory reach has no limits. Including formulas or classes they define in the domain of quantifiers just climbs higher in that direction. Little relevance to mainstream math.

Takeaway: a Way to Handle

5. Separating formulas from their external parameters in math objects allows restricting formula-related information from parameters.

6. Complexity theory allows to formalize this, justify for the "external data", and use for radical simplification of math foundations.

7. What is left out ? The "Logics" sets, related to infinite hierarchies of formulas, such as, e.g., "The set of all true sentences of Arithmetic". Those should be subject of math **foundations**. Theories cannot include their own foundations.

IP has other powerful applications.
They are unrelated to Set Theory,
so, I will just mention a couple.

Some More IP Applications

Foundations of Probability Theory. Paradoxes in its application led to the concept of K-ML Randomness $\mathbf{R}$. **IP** clarifies its use: Any $S \subset \Omega$ has measure 0 **if and only if** all K-ML-random sequences in S have infinite information about a common sequence σ.

Goedel's Theorem Loophole. Goedel writes:

> "It is not at all excluded by the negative results mentioned earlier that nevertheless every clearly posed mathematical yes-or-no question is solvable in this way. For it is just this becoming evident of more and more new axioms on the basis of the meaning of the primitive notions that a machine cannot imitate."

No way ! Let a predicate P extend the Peano Arithmetic "proven/refuted" partial predicate to all n-bit sentences. Then $\mathbf{I}(P:r) > n - \log n$ for $r = \min \mathbf{R}_0$. No way to obtain such P by either formal OR any informal methods !

Appendix: ZFC Axioms

1. **Membership chains**. (1b anti-dual to 1a) :

1a. Infinity (a set with no source) :

$$\boxed{\exists S, s \in S\, \forall x \in S\, \exists y \in S(x \in y)}$$

1b. Foundation (sinks in all sets) :

$$\boxed{\neg\, \exists S, s \in S\, \forall x \in S\, \exists y \in S(y \in x)}$$

2. **Formulas define sets (content-defined)** :

2a. Extensionality : $\boxed{x \supset y \supset x \in t \Rightarrow y \in t}$

2b. Replacement ($R_c(X) \overset{\text{df}}{=} \{y \colon \exists x \in X\, R_c(x,y)\}$) :

$$\boxed{(\forall x\, \exists Y \supset R_c(\{x\})) \Rightarrow \forall X\, \exists Y \supset R_c(X) \supset Y}$$

3. **Function Inverses** ($f^{-1} \overset{\text{df}}{=} \{g \subset f^T \colon f \circ g \circ f = f\}$) :

3a. Power set (f^{-1} is a set, take $h := f^T$) :

$$\boxed{\forall h\, \exists G\, \forall g \subset h\, (g \in G)}$$

3b. Choice (f^{-1} is not empty): $\boxed{\forall f\, \exists\, g \in f^{-1}}$

To Modify ZFC

Math objects are classes of sets defined by math formulas with external parameters. Collections of objects are treated as collections of those parameters (sets). Sets respect axioms, modified as follows:

1. Foundation and Extensionality are extended (as families) to classes of sets;
2. Replacement is restricted to computable R;
3. Add "All sets countable."; drop Power Set;
4. Add **IP**; Add $\mathbf{P}\chi$, perhaps strengthened;
5. Some change with the Choice. (May be dropping it, or adding to the language a postulated (named, not described) class injecting reals into countable v.Neumann ordinals (continuum hypothesis implied).)

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