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ON STORAGE CAPACITY FOR ALGORITHMS

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Problems connected with the computational complexity of functions have been investigated in many papers. We single out [1], [2], in which compression and speedup theorems were proved that received broad repute.

A function for which there is an optimal computing algorithm is constructed in the compression theorem, and a computable function for which there is no optimal algorithm (i.e. every algorithm can be substantially improved), in the speedup theorem. These two theorems seem to be contradictory; however, A. N. Kolmogorov observed that they are both special cases of the following more general assertion:

Let $s_i(x)$ *be a computable decreasing sequence of monotone general recursive functions computable on a Turing machine with the utilization of storage capacity equal to their value. Then a general recursive function* $f(x)$ *exists for which for every* i *an algorithm computing it exists which utilizes storage capacity equal to* $s_i(x)$, *and for every algorithm computing* f *there is an* i *such that the storage capacity utilized by that algorithm is of order not less than* $s_i(x)$.

This idea induced the author to investigate what properties in general a signaling class for computing an arbitrary given general recursive function must possess. Quite complete results on this subject are obtained in the present paper. They imply in particular the compression and speedup theorems and the conjecture of Kolmogorov which is set forth above as elementary special cases.

We shall consider Turing machines with different alphabets on tape. If the cardinality of an alphabet on tape equals k, natural numbers are considered to be in k-adic notation. We call a partially recursive function $f(x)$ for which an algorithm exists which computes it with the use of not more than $f(x)$ sectors of tape an *elementary function*. (We call that algorithm an *elementary algorithm*.) We assume by convention that an elementary function equals ∞ outside of its domain of definition.

Let a general recursive function $f(x)$ be given. Consider the class M_f of functions $s(x)$ which prescribe the volume of tape utilized by algorithms computing $f(x)$ in its domain of definition. We call this class the *class of signaling capacities for* f. It will possess the following obvious properties:

a) All functions belonging to M_f are elementary.

b) If $s_1 \in M_f$, s_2 is an elementary function and $s_2 \geq s_1/2$, then $s_2 \in M_f$.

c) If $s_1, s_2 \in M_f$, then $\min(s_1, s_2) \in M_f$.

d) A general recursive function $s_1 \in M_f$ exists.

e) The class M_f is anti-enumerable, i.e. it is generated by some set of elemen-

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tary algorithms whose complement is denumerable.

We call every class of functions possessing properties a)—e) *canonical*.

Theorem 1*. *The class of signaling capacities of any function is canonical (which is almost trivial), and for any canonical class a general recursive function exists which takes the values* 0 *and* 1 *and for which it will be the class of signaling capacities.*

The following theorem shows that canonical classes are very simply constituted.

Theorem 2 (on the greatest lower bound). *Let $g(x)$ be an arbitrary general recursive function and M_g be its class of signaling capacities.*

Then a general recursive function $f(x)$ exists such that M_g coincides with the class of all elementary functions of order not less than $f(x)$.

The converse is obviously true (since for any general recursive function $f(x)$ the class of elementary functions of order not less than $f(x)$ will be canonical).

Remark 1. By virtue of Blum's compression theorem [2], the function f in Theorem 2 cannot always be chosen elementary; however, it can always be chosen semielementary, i.e. such that an algorithm $A(x, n)$ exists for it which for every x and $n \geq f(x)$ delivers up $f(x)$, with storage volume not greater than n being utilized.

More general problems exist than the problem of computing a given function. Let us examine one of these generalizations. Let a denumerable set A of pairs of numbers (x, y) be given. The problem connected with them is: for a given x obtain some y such that $(x, y) \notin A$. We call a problem of this kind a generalized problem if it can be solved by at least some general recursive algorithm. The class of signaling capacities for the generalized problem A is defined analogously to the preceding. It is interesting that these classes will be the same as for ordinary problems of the predicate calculus.

Remark 2. The class of signaling capacities of an arbitrary generalized problem will be canonical.

All results are easily extended to the case of partially recursive functions. Let A be an enumerable set. Let us change items d) and b) of the definition of a canonical set in the following way:

b′) If $s_1 \in M_f$, s_2 is an elementary function and $s_2(x) \geq s_1(x)/2$ for all $x \in A$ except a finite number, then $s_2 \in M_f$.

d′) A function s_1 exists with domain of definition A, belonging to the class M_f.

We call such a class *partially canonical* with domain of definition A.

Remark 3. Theorems 1 and 2 remain valid under substitution of the words "partially recursive" for "general recursive function" and "partially canonical" for "canonical class", with the same domain of definition.

The latter remark implies in particular the solution of Slisenko's problem concerning whether the speedup theorem holds for the problem of enumerating creative sets.

Let an algorithm g enumerate some set A. We call a function $s(x)$ which equals the length of tape utilized by the algorithm for delivering up all elements of A less than x a *monotone signaling enumeration*. (If A is unsolvable, $s(x)$ will be greater than any general recursive function.)

Corollary 1. *A creative set A exists for which there is a monotone signaling enumeration which is optimal with respect to order. Also, for every general recursive $g(x)$ a creative set A_g exists which has for every monotone signaling enumeration $s_1(x)$ another signaling $s_2(x)$ such that $s_1(x) \geq g(s_2(x))$.*

All results clearly hold not only for the storage volume of a Turing machine but also for any other type of signaling in the sense of [2] to within, it is true, not order but some other general recursive function.

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