Fundamentals of Computing

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Abstract

These are notes for the course CS-172 I first taught in the Fall 1986 at UC Berkeley and subsequently at Boston University. The goal was to introduce students to basic concepts of Theory of Computation and to provoke their interest in further study. Model-dependent effects were systematically ignored. Concrete computational problems were considered only as illustrations of general principles.

The notes are skeletal: they do have (terse) proofs, but exercises, references, intuitive comments, examples are missing or inadequate. The notes can be used for designing a course or by students who want to refresh the known material or are bright and have access to an instructor for questions.
Each subsection takes about a week of the course. Versions of these notes appeared in [Levin 91].

Acknowledgments. I am grateful to the University of California at Berkeley, its MacKey Professorship fund and Manuel Blum who made possible for me to teach this course. The opportunity to attend lectures of M. Blum and Richard Karp and many ideas of my colleagues at BU and MIT were very beneficial for my lectures. I am also grateful to the California Institute of Technology for a semester with light teaching load in a stimulating environment enabling me to rewrite the students' notes. NSF grants #DCR-8304498, DCR-8607492, CCR-9015276 also supported the work. And most of all I am grateful to the students [see 6.3] who not only have originally written these notes, but also influenced the lectures a lot by providing very intelligent reactions and criticism.

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 Last revised: March 3, 2025.

Part I
Basics

1 Deterministic Models; Polynomial Time & Church's Thesis

Sections 1,2 study deterministic computations. Non-deterministic aspects of computations (inputs, interaction, errors, randomization, etc.) are crucial and challenging in advanced theory and practice. Defining them as an extension of deterministic computations is simple. The latter, however, while simpler conceptually, require elaborate models for definition. These models may be sophisticated if we need a precise measurement of all required resources. However, if we only need to define what is computable and get a very rough magnitude of the needed resources, all reasonable models turn out equivalent, even to the simplest ones. We will pay significant attention to this surprising and important fact. The simplest models are most useful for proving negative results and the strongest ones for positive results.

We start with terminology common to all models, gradually making it more specific to those we actually study. We represent ***computations*** as graphs: the edges reflect various relations between nodes (***events***). Nodes, edges have attributes: labels, states, colors, parameters, etc. (affecting the computation or its analysis). ***Causal*** edges run from each event to all events essential for its occurrence or attributes. They form a directed acyclic graph (though cycles may be added artificially to mark the external input parts of the computation).

We will study only ***synchronous*** computations. Their nodes have a ***time*** parameter. It reflects logical steps, not necessarily a precise value of any physical clock. Causal edges only span short (typically, ≤ 3 moments) time intervals. One event among the causes of a node is called its ***parent***. ***Pointer*** edges connect the parent of each event to all its other possible causes and reflect connections that allow simultaneous events to interact and have a joint effect. Pointers with the same source have different labels. The (labeled) subgraph of events/edges at a given time is an instant memory ***configuration*** of the model.

Each non-terminal configuration has ***active*** nodes/edges around which it may change. The models with only a small active area at any step of the computation are ***sequential***. Others are called ***parallel***.

Complexity. We use the following measures of computing resources of a machine A on input x:

Time: The greatest depth $D_{A(x)}$ of causal chains is the number of computation steps. The volume $V_{A(x)}$ is the combined number of active edges during all steps. Time $T_{A(x)}$ is used (depending on the context) as either depth or volume, which are close for sequential models. Note that time complexity is robust only up to a constant factor: a machine can be modified into a new one with a larger alphabet of labels, representing several locations in one. It would produce identical results in a fraction of time and space (provided that the time limits suffice for transforming the input and output into the other alphabet).

Space: $S_{A(x)}$ or $S_A(x)$ of a synchronous computation is the greatest (over time) size of its configurations. Sometimes excluded are nodes/edges unchanged since the input.

Growth Rates (typically expressed as functions of bit length $n = \|x, y\|$ of input/output x/y):

O, Ω: $f(n) = O(g(n))$[1] $\iff$ $g(n) = \Omega(f(n)) \iff \sup_n \frac{f(n)}{g(n)} < \infty$.

$o, \omega : f(n) = o(g(n)) \iff g(n) = \omega(f(n)) \iff \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$.

$\Theta : f(n) = \Theta(g(n)) \iff (f(n) = O(g(n))$ and $g(n) = O(f(n)))$.

Here are a few examples of frequently appearing growth rates: negligible $(\log n)^{O(1)}$; moderate $n^{\Theta(1)}$ (called polynomial or P, like in P-time); infeasible: $2^{n^{\Omega(1)}}$, also $n! = (n/e)^n\sqrt{\pi(2n+1/3)+\varepsilon/n}$, $\varepsilon \in [0,\ .1]$.[2]

The reason for ruling out exponential (and neglecting logarithmic) rates is that the known Universe is too small to accommodate exponents. Its radius is about 46.5 giga-light-years $\sim 2^{204}$ Plank units. A system of $\gg R^{1.5}$ atoms packed in R Plank Units radius collapses rapidly, be it Universe-sized or a neutron star. So the number of atoms is $< 2^{306} \ll 4^{4^4} \ll 5!!$.

[1] This is a customary but somewhat misleading notation. The clear notations would be like $f(n) \in O(g(n))$

[2] A rougher estimate follows by computing $\ln n! = t\ln(t/e)|_{t=1.5}^{n+.5} + O(1)$ using that $|\sum_{i=2}^{n} g(i) - \int_{1.5}^{n+.5} g(t)\,\mathbf{d}t|$ is bounded by the total variation v of $g'/8$. So for monotone $g'(t) = \ln'(t) = 1/t$ the $O(1)$ is $< v < 1/12$.

1.1 Rigid Models

Rigid computations have another node parameter: ***location*** or ***cell***. Combined with time, it designates the event uniquely. Locations have ***structure*** or ***proximity*** edges between them. They (or their short chains) indicate all neighbors of a node to which pointers may be directed.

Cellular Automata (CA)

CA are a parallel rigid model. Its sequential restriction is the ***Turing Machine (TM)***. The configuration of CA is a (possibly multi-dimensional) grid with a finite, independent of the grid size, alphabet of ***states*** to label the events. The states include, among other values, pointers to the grid neighbors. At each step of the computation, the state of each cell can change as prescribed by a ***transition*** function (also called program) applied to the previous states of the cell and its pointed-to neighbors. The initial state of the cells is the input for the CA. All subsequent states are determined by the transition function.

An example of a possible application of CA is a VLSI (very large scale integration) chip represented as a grid of cells connected by wires (chains of cells) of different lengths. The propagation of signals along the wires is simulated by changing the state of the wire cells step by step. The clock interval can be set to the time the signals propagate through the longest wire. This way the delays affect the simulation implicitly.

An example: the Game of Life (GL). GL is a plane grid of cells, each holds a 1-bit state (dead/alive) and pointers to the 8 adjacent cells. A cell remains dead or alive if the number i of its live neighbors is 2. It becomes (or stays) alive if i=3. In all other cases it dies (of overcrowding or loneliness) or stays dead.

A ***simulation*** of a machine M_1 by M_2 is a correspondence between memory configurations of M_1 and M_2 which is preserved during the computation (may be with some time dilation). Such constructions show that the computation of M_1 on any input x can be performed by M_2 as well. GL can simulate any CA (see a sketch of an ingenious proof in the last section of [Berlekamp, Conway, Guy 82]) in this formal sense:

We fix space and time periods a, b. Cells (i, j) of GL are mapped to cell $(\lfloor i/a \rfloor, \lfloor j/a \rfloor)$ of CA M (compressing $a \times a$ blocks). We represent cell states of M by states of $a \times a$ blocks of GL. This correspondence is preserved after any number t steps of M and bt steps of GL regardless of the starting configuration.

Turing Machines (TMs)

TM is a *minimal CA*. Its configuration – ***tape*** – is an infinite to the right chain of cells.
Each state of a cell has a pointer to one of the cell's two adjacent neighbors. No adjacent cells can both point away from each other. Only the two cells pointing at each other are ***active***, i.e. can change state.
The cell that just turned its pointer is the TM's moving head working on the tape symbol - its target.
The input is an array of non-blank cells (only one is rightward) followed by blanks at the right.

Another type of CA represents a TM A with several non-communicating heads. At most $O(1)$ heads fit in a cell. They can vanish, split, or merge only in the first cell (which, thus, controls the number of active cells). The input x makes an unchangeable "ink" part of each cell's state. The rest of the cell's state is in "pencil" and can be changed by A. The computation halts when all heads drop off. The output $A(x)$ is the pencil part of the tape's state. This model has convenient theoretical features. E.g. with linear (in T) number ($\|p\|^2 T$) of state changes (volume) one can solve the ***Bounded Halting Problem*** $H(p, x, T)$: find out whether the machine with a program p stops on an input x within volume T of computation (see 2.3).

Exercise: Find a method to transform any given multi-head TM A into another one B such that the value of the output of $B(x)$ (as a binary integer) and the volumes of computation of $A(x)$ and of $B(x)$ are all equal within a constant factor (for all inputs x). **Hint:** B-cells may have a field to simulate A and maintain (in other fields) two binary counters h (with $\Theta(1)$ density of heads) for the number of heads of A and v for A's volume. Their least significant digits are at the leftmost cell. h adds its most significant digit to the same position in v at each step of A. To move the carry 1 on v a head is borrowed from h. These 1-heads move right in v till they empty their carry into its 0 digit. Then empty 0-heads move back to h in a separate field/track, possibly first continuing right to find a free slot in this return track. (The heads area in v extends to k-th cell only by dropping the carry there, with frequency $O(2^{-k})$. Then it shrinks to $O(1)$ in $O(k)$ steps since heads enter it slower than they move away.) Borrowed or returned heads make low or high head-density areas in h which shift left until absorbed at the leftmost cell.

1.2 Pointer Machines

The memory configuration of a ***Pointer Machine (PM)***, called ***pointer graph***, is a finite directed labeled multigraph. One node R is marked as ***root*** and has directed paths to all nodes. Nodes can ***see*** and change the configuration of their out-neighborhood of constant (2 suffices) depth. Edges (***pointers***) are labeled with ***colors*** from a finite alphabet common to all graphs handled by a given program. The pointers coming out of a node must have different colors (which bounds the outdegree). Some colors are designated as ***working*** and not used in inputs/outputs. One of them is called ***active***, as also are pointers carrying it and nodes seeing them. Active pointers must have inverses, form a tree to the root, and can be dropped only in leaves.

All active nodes each step execute an identical ***program***. At its first ***pulling*** stage, node A acquires copies of all pointers of its children using "composite" colors: e.g., for a two-pointer path (A, B, C) colored x, y, the new pointer (A, C) is colored xy, or an existing z-colored pointer (A, C) is recolored $\{z, xy\}$. A also spawns a new node with pointers to and from it. Next, A transforms the colors of its set of pointers, drops the pointers left with composite colors, and vanishes if no pointers are left. Nodes with no path from the root are forever invisible and considered dropped. The computation is initiated by inserting an active loop-edge into the root. When no active pointers remain, the graph, with all working colors dropped, is the output.

Exercise: Design a PM transforming the input graph into the same one with two extra pointers from each node: to its parent in a BFS spanning tree and to the root. Hint: Nodes with no path *to* the root can never be activated. but can be copied with pointers, copies connected to the root, the original input removed.

PM can be ***parallel, PPM*** [Barzdin', Kalnin's 74] or ***sequential, SPM***. SPM differ in that only pointers to the root, their sources, and nodes that have pointers with inverses to these sources can be active.

A ***Kolmogorov*** or ***Kolmogorov-Uspenskii*** Machine (KM) [Kolmogorov, Uspenskii 58], is a special case of Pointer Machine [Schoenhage 80] with the restriction that all pointers have inverses. This implies the bounded in/out-degree of the graph which we further assume to be constant.

Fixed Connection Machine (FCM) is a variant of the PKM with the restriction that pointers once created *cannot* be removed, only re-colored. So when the memory limits are reached, the pointer structure freezes, and the computation can be continued only by changing the colors of the pointers.

PPM is the most powerful model we consider: it can simulate the others in the same space/time. E.g., cellular automata make a simple special case of a PPM which restricts the Pointer Graph to be a grid.

Exercise. Design a machine of each model (TM, CA, KM, PPM) which determines if an input string x has a form ww, $w \in \{a, b\}^*$. Analyze time (depth) and space. KM/PPM takes input x in the form of colors of edges in a chain of nodes, with root linked to both ends. The PPM nodes also have pointers to the root. Below are hints for TM,SPM,CA. The space is $O(\|x\|)$ in all three cases.

Turing and Pointer Machines. TM first finds the middle of ww by capitalizing the letters at both ends one by one. Then it compares letter by letter the two halves, lowering their case. The complexity is: $T(x) = O(\|x\|^2)$. SPM acts similarly, except that the root keeps and updates the pointers to the borders between the upper and lower case substrings. This allows constant time access to these borders. So, $T(x)=O(\|x\|)$.

Cellular Automata. The computation starts with the leftmost cell sending right two signals. Reaching the end the first signal turns back. The second signal propagates three times slower, so they meet in the middle of ww and disappear. While alive, the second signal copies the input field i of each cell into a special field c. The c symbols will try to move right whenever the next cell's c field is blank. So the chain of these symbols alternating with blanks will start moving right from the middle of ww. Upon reaching the end they will push the blanks out and pack themselves back into a copy of the left half of ww shifted right. When a c symbol does not have a blank at the right to move to, it compares itself with the i field of the same cell. If they differ, a signal is generated which halts all activity and rejects x. If all comparisons are successful, the last c generates the accepting signal. The depth is: $T(x) = O(\|x\|)$.

1.3 Simulation

We have considered several models of computation. We will see now how the simplest of them – Turing Machine – can simulate all others: these powerful machines can compute no more functions than TM.

Church-Turing Thesis is a generalization of this conclusion: TMs can compute every function computable in any thinkable physical model of computation. This is not a math theorem because the notion of model is not formally specified. But the long history of studying ways to design real and ideal computing devices makes it very convincing. Moreover, this Thesis has a stronger ***Polynomial Time*** version which bounds the volume of computation required by that TM simulation by a polynomial of the volume used by the other models. Both forms of the Thesis play a significant role in foundations of Computer Science.

PKM Simulation of PPM. For convenience, we assume all PPM nodes have pointers to root. PPM configuration is represented in PKM with extra colors l, r, u used in a u-colored binary tree added to each node X so that all (unlimited in number) PPM pointers to X are reconnected to its leaves, and inverses, colored l, r, added to all pointers. The number of pointers increases at most 4 times. To simulate PPM, X gets a binary ***name*** formed by the l, r colors on its path through the root tree, and broadcasts it down its own tree. For pulling stage X extends its tree to double depth and merges (with combined colors) its own pointers to nodes with identical names. Then X re-colors its pointers as PPM program requires and rebalances its tree. This simulation of a PPM step takes polylogarithmic parallel time.

TM Simulation of PKM. We assume the PKM keeps a constant degree and a roughly balanced root tree (to yield short node names as described above). TM tape reflects its configuration as the list of all pointers sorted by source name, then by color. The TM's transition table reflects the PKM program. To simulate PKM's pulling stage TM creates a copy of each pointer and sorts copies by their sinks. Now each pointer, located at source, has its copy near its sink. So both components of 2-pointer paths are nearby: the special double-colored pointers can be created and moved to their sources by resorting on the source names. The re-coloring stage is straightforward, as all relevant pointers having the same source are located together. Once the root has no active pointers, the Turing machine stops and its tape represents the PKM output. If a PPM computes a function $f(x)$ in $t(x)$ steps, using $s(x)$ nodes, the simulating TM uses space $S = O(s \log s)$, ($O(\log s)$ bits for each of $O(s)$ pointers) and time $T = O(S^2)t$, as TM sorting takes quadratic time.

Squaring matters ! TM cannot outperform Bubble Sort. Is its quadratic overhead a big deal? In a short time all silicon gates on your PC run, say, $X{=}10^{23}{\sim}2^{2^{6.25}}$ clock cycles combined. Silicon parameters double almost annually. Decades may bring micron-thin things that can sail sunlight in space in clouds of great computing and physical (light beam) power. Centuries may turn them into a Dyson Sphere enveloping the solar system. Still, the power of such an ultimate computer is limited by the number of photons the Sun emits per second: $Y{\sim}2^{2^{7.25}}{=}X^2$. Giga-years may turn much of the known universe into a computer, but its might is still limited by its total entropy $2^{2^{8.25}}{=}Y^2$.

Faster PPM Simulations. Parallel Bubble-Sort on CA or Merge-Sort on sequential FCM take nearly linear time. Parallel FCM can do much better [Ofman 65]. It represents and updates pointer graphs as the above TM. All steps are straightforward to do locally in parallel polylog time except sorting of pointers. We need to create a fixed connection sorting network. Sophisticated networks sort arbitrary arrays of n integers in $O(\log n)$ parallel steps. We need only a simpler polylog method. Merge-Sort splits an array of two or more entries in two halves and sorts each recursively. Batcher-Merge combines two sorted lists in $O(\log n)$ steps.

Batcher Merge. A ***bitonic cycle*** is the combination of two sorted arrays (one may be shorter), connected by max-to-max and min-to-min entries. Entries in a contiguous half (***high-half***) of the cycle are $\geq$ than all entries in the other (***low***) half. Each half (with its ends connected) forms a bitonic cycle itself.

Shuffle Exchange graph links nodes in a 2^k-nodes array to their ***flips*** and ***shits***. The flip flips the highest bit of a node's address; the ***shift*** cycle-shifts that bit to the end, or switches 0^k address with 1^k.

We merge-sort two sorted arrays given as a bitonic cycle on such a graph as follows.
Comparing each entry with its flip (half-a-cycle away), and switching if wrongly ordered, fits the high and low halves into respectively the first and last halves of the array. (This rotates the array, and then its left and right halves.) We do so for each half recursively (decrementing k via graph's shift edges).

2 Universal Algorithm; Diagonal Results

2.1 Universal Turing Machine

The first computers were hardware-programmable. To change the function computed, one had to reconnect the wires or even build a new computer. John von Neumann suggested using Turing's Universal Algorithm. The function computed can be then specified by just giving its description (program) as part of the input rather than by changing the hardware. This was a radical idea, since in the classical mathematics universal functions do not exist (as we will see in Sec. 2.2).

Let R be the class of all TM-computable functions: total (defined for all inputs) and partial (which may diverge). Surprisingly, there is a universal function u in R. For any Turing Machine M that computes $f{\in}R$ in time T and space S, u uses a program m of length c listing the commands and initial head state of M. Then $u(mx)$ simulates $M(x)$ in time c^2T and space $S{+}c$. It operates in cycles, each simulating one step of $M(x)$. After i steps of $M(x)$, let s_i be the head's state, l_i be the left from it part of the tape; r_i be the rest of the tape. After i cycles $u(mx)$ has the tape configuration $t_i = l_i m s_i r_i$ and looks up m to find the command corresponding to the state s_i and the first symbol of r_i. It modifies t_i accordingly. When $M(x)$ halts, $u(mx)$ erases the (penciled) ms_i from the tape and halts too. Universal Multi-head TM works similarly but can also determine in time $O(t(x))$ whether it halts in t steps (given $x, t(x)$ and an appropriate program).

Exercise. Design a universal multi-head TM with a constant factor overhead.
Hint: When heads split or merge in the first cell, the room u needs for their programs
creates sparse or dense content regions that propagate right (sparse faster).

We now describe in detail a simpler but slower universal [Ikeno 58] TM U. It simulates any other TM M that uses only $0, 1$ bits on the tape. (Other alphabets are encoded as bit strings for this.) M lacks the blank symbol that usually marks the end of input. Thus input needs to be given in some prefixless form, e.g., with a padding that encodes input length $l = \|x\|$ as a binary number preceded by $2\|l\|$ zeros. In the cells carrying this padding, two counters are initiated that monitor the distance to both ends of the used part of M's tape (initially the input). M's head moving on the tape pulls these counters along and keeps them updated. When the right end of the used tape is reached, any subsequent characters are treated as blanks.

U has 11 states + 6 symbols; see its transition table at the right. It shows the states and tape digits only when changed, except that the prime is always shown. The head is on the tape: lower case states look left, upper case – right. The external choice, halt, etc. commands are special states for M; for U they are shown as "A/B" or =. U works in cycles, simulating one transition of M each. The tape is infinite to the right (the leftward head in the leftmost cell halts). It consist of 0/1 segments, each preceded with a *. Some symbols are primed. The rightmost infinite part of one or two segments is always a copy of M's tape. The other segments describe one transition each: a command $(s, b) \to (s', b', d)$ for M to change state s to s', tape bit b to b' and turn left or right.

	1	0	*	1'	0'	*'
A	f	f	e0			
B	F	F	e1			
f,F	b*	a*	F			c
D	d'	–				e'
E	=	–	'	'	'	e'
d	'	'		'	'	D
b	'	'	'		a'	D
a	'	'	'	b'	F	E'
c	'	'		=	F	E'
e	'	'	'	B	A	A/B

The transition segments are sorted in order of (s, b) and never change, except for priming. Each transition is represented as $*Sdb$, where b is the bit to write, d the direction $R{=}0/L{=}1$ to turn. S points to the next state represented as 1^k, if it is k segments to the left, or 0^k (if to the right). Each cycle starts with U's head in state F or f, located at the site of M's head. Primed are the digits of S in the prior command and all digits to their left. An example of the configuration: $*'0'0'0'1'0' *' 0'0'0'0'01 * 011 * \ldots * 00$ $\boxed{\text{head}}$ $00\ldots$

U first reads the bit of an M's cell changing the state from F or f to a/b, puts a * there, moves left to the primed state segment S, finds from it the command segment and moves there. It does this by repeatedly priming nearest unprimed * and 1s of S (or unpriming 0s) while alternating the states c/F or d/D. When S is exhausted, the target segment $\|S\| + b$ stars away is reached. Then U reads (changing state from e to A or B) the rightmost symbol b' of the command, copies it at the * in the M area, goes back, reads the next symbol d, returns to the just overwritten (and first unprimed) cell of M area and turns left or right. As CA, M and U have in each cell three standard bits: present d and previous d' pointer directions and a "content" bit to store M's symbol. In addition U needs just one "trit" of its own!

2.2 Uncomputability; Goedel Theorem

Universal and Complete Functions

Notations: Let us choose a special mark and after its k-th occurrence, break any string x into $\text{Prefix}_k(x)$ and $\text{Suffix}_k(x)$. Let $f^+(x)$ be $f(\text{Prefix}_k(x)\ x)$ and $f^-(x)$ be $f(\text{Suffix}_k(x))$. We say u ***k-simulates*** f iff for some $p = \text{Prefix}_k(p)$ and all s, $u(ps) = f(s)$. The prefix can be intuitively viewed as a program which simulating function u applies to the suffix (input). We also consider a symmetric variant of relation "k-simulate" which makes some proofs easier. Namely, u ***k-intersects*** f iff $u(ps) = f(ps)$ for some $\text{prefix}_k\ p$ and all s. E.g., length preserving functions can intersect but cannot simulate one another.

We call ***universal*** for a class F, any u which k-simulates all functions in F for a fixed k. When F contains f^-, f^+ for each $f \in F$, universality is equivalent to (or implies, if only $f^+ \in F$) ***completeness***: u k-intersects all $f \in F$. Indeed, u k-simulates f iff it k-intersects f^-; u $2k$-intersects f if it k-simulates f^+.

Exercise: Describe **explicitly** a function, **complete** for the class of all *linear* (e.g., $5x$ or $23x$) functions.

A ***negation*** of a (partial or total) function f is the total predicate $\neg f$ which yields 1 iff $f(x)$=0 and yields 0 otherwise. Obviously, no closed under negation class of functions contains a complete one. So, there is no universal function in the class of all (computable or not) predicates. This is the well known Cantor Theorem that the set of all sets of strings (as well as the sets of all functions, reals etc.) is not countable.

Goedel's Theorem

There is no complete function among the **total** computable ones, as this class is closed under negation. So the universal in R function u (and $u_2 = (u \bmod 2)$) has no total computable extensions.

Formal proof systems are computable functions $A(P)$ which check if P is an acceptable proof and output the proven statement. $\vdash s$ means $s = A(P)$ for some P. A is ***rich*** iff it allows computable translations s_x of statements "$u_2(x) = 0$", provable whenever true, and refutable ($\vdash \neg s_x$), whenever $u_2(x) = 1$. A is ***consistent*** iff **at most** one of any such pair $s_x, \neg s_x$ is provable, and ***complete*** iff **at least** one of them always (even when $u(x)$ diverges) is. Rich consistent and complete formal systems cannot exist, since they would provide an obvious total extension u_A of u_2 (by exhaustive search for P to prove or refute s_x). This is the famous Goedel's Theorem – one of the shocking surprises of the 20th century science. (Here A is any extension of the formal Peano Arithmetic; we skip the details of its formalization and proof of richness.)[3]

Computable Functions. Another byproduct is that the Halting (of $u(x)$) Problem would yield a total extension of u and, thus, is not computable. This is the source of many other uncomputability results. Another source is an elegant ***Fixed Point*** Theorem by S. Kleene: any total computable transformation A of programs (prefixes) maps some program into an equivalent one. Indeed, the complete/universal $u(ps)$ intersects computable $u(A(p)s)$. This implies, e.g., Rice theorem: the only computable invariant (i.e. the same on programs computing the same functions) property of programs is constant (**exercise**).

Computable (partial and total) functions are also called recursive (due to an alternative definition). Their ranges (and, equivalently, domains) are called (computably) ***enumerable*** or ***c.e.*** sets. An c.e. set with an c.e. complement is called computable (as is its yes/no characteristic function) or ***decidable***. A function is computable iff its graph is c.e. An c.e. graph of a total function is computable. Each infinite c.e. set is the range of an injective total computable function ("enumerating" it, hence the name c.e.).

We can reduce membership problem of a set A to the one of a set B by finding a computable function f s.t. $x \in A \iff f(x) \in B$. Then A is called ***m-*** (or ***many-to-1-***) ***reducible*** to B. A more complex ***Turing*** reduction is given by an algorithm which, starting from input x, interacts with B by generating strings s and receiving answers to $s \in ?B$ questions. Eventually it stops and tells if $x \in A$. c.e. sets (like Halting Problem) to which all c.e. sets can be m-reduced are called c.e.-complete. One can show a set c.e.-complete (and, thus, undecidable) by reducing the Halting Problem to it. So Ju.Matijasevich proved c.e.-completeness of Diophantine Equations Problem: given a multivariate polynomial of degree 4 with integer coefficients, find if it has integer roots. The above (and related) concepts and facts are broadly used in Theory of Algorithms and should be learned from any standard text, e.g., [Rogers 67].

[3] A closer look at this proof reveals another famous Goedel theorem: Consistency C of A (expressible in A as divergence of the search for contradictions) is itself an example of unprovable $\neg s_x$. Indeed, u_2 intersects $1-u_A$ for some prefix a. C implies that u_A extends u_2 and, thus, $u_2(a), u_A(a)$ both diverge. So, $C \Rightarrow \neg s_a$. This proof can be formalized in Peano Arithmetic, thus $\vdash C \Rightarrow \vdash \neg s_a$. But $\vdash \neg s_a$ implies $u_A(a)$ converges, so $\vdash C$ contradicts C: Consistency of A is provable in A if and only if false !

2.3 Intractability; Compression and Speed-up Theorems

The ***t-restriction*** u_t of u aborts and outputs 1 if $u(x)$ does not halt within $t(x)$ steps, i.e. u_t computes the ***t-Bounded Halting Problem (t-BHP)***. It remains complete for the closed under negation class of functions computable in $o(t(x))$ steps. ($O(\|p\|^2)$ overhead is absorbed by $o(1)$ and padding p.) So, u_t is not in the class, i.e. cannot be computed in time $o(t(x))$ [Tseitin 56]. (And neither can be any function agreeing with t-BHP on a ***dense*** (i.e. having strings with each prefix) subset.) E.g. $2^{\|x\|}$-BHP requires exponential time.

However for some trivial input programs the BHT can obviously be answered by a fast algorithm. The following theorem provides another function $P_f(x)$ (which can be made a predicate) for which there is only a finite number of such trivial inputs. We state the theorem for the volume of computation of Multi-Head Turing Machine. It can be reformulated in terms of time of Pointer Machine and space (or, with smaller accuracy, time) of regular Turing Machine.

Definition: A function $f(x)$ is ***constructible*** if it can be computed in volume $V(x) = O(f(x))$.

Here are two examples: $2^{\|x\|}$ is constructible, as $V(x) = O(\|x\| \log \|x\|) \ll 2^{\|x\|}$.
Yet, $2^{\|x\|} + h(x)$, where $h(x)$ is 0 or 1, depending on whether $U(x)$ halts within $3^{\|x\|}$ steps, is not.

Compression Theorem [Rabin 59]. For any constructible function f, there exists a function P_f such that for all functions t, the following two statements are equivalent:

1. There exists an algorithm A such that $A(x)$ computes $P_f(x)$ in volume $t(x)$ for all inputs x.
2. t is constructible *and* $f(x) = O(t(x))$.

Proof. Let ***t-bounded Kolmogorov Complexity*** $K_t(i|x)$ of i given x be the length of the shortest program p for the Universal Multi-Head Turing Machine transforming x into i with $< t$ volume of computation. Let $P_f(x)$ be the smallest i, with $2K_t(i|x) > \log(f(x)|t)$ for all t. P_f is computed in volume f by generating all i of low complexity, sorting them and taking the first missing. It satisfies the Theorem, since computing $i{=}P_f(x)$ faster would violate the complexity bound defining it. (Some extra efforts can make P Boolean.) □

Speed-up Theorem [Blum 67]. There exists a total computable predicate P such that for any algorithm computing $P(x)$ in volume $t(x)$, there exists another algorithm doing it in volume $O(\log t(x))$.

Though stated here for exponential speed-up, this theorem remains true with log replaced by any computable unbounded monotone function. In other words, there is no even nearly optimal algorithm to compute P.

The general case. So, the complexity of some predicates P cannot be characterized by a single constructible function f, as in Compression Theorem. However, the Compression Theorem remains true (with harder proof) if the requirement that f is constructible is dropped (replaced with being computable).[4]
In this form it is general enough so that every computable predicate (or function) P satisfies the statement of the theorem with an appropriate computable function f. There is no contradiction with Blum's Speed-up, since the complexity f (not constructible itself) cannot be reached. See a review in [Seiferas, Meyer 95].

[4]The proof stands if constructibility of f is weakened to being ***semi-constructible***, i.e. one with an algorithm $A(n, x)$ running in volume $O(n)$ and such that $A(n, x){=}f(x)$ if $n{>}f(x)$. The sets of programs t whose volumes (where finite) satisfy either (1) or (2) of the Theorem (for computable P, f) are in Σ_2^0 (i.e. defined with 2 quantifiers). Both generate monotone classes of constructible functions closed under $\min(t_1, t_2)/2$. Then any such class is shown to be the $\Omega(f)$ for some **semi-constructible** f.

3 Games; Alternation; Exhaustive Search; Time vs. Space

3.1 How to Win

In this section we consider a more interesting *provably* intractable problem: playing games with full information, two players and zero sum. We will see that even for some simple games there cannot be a much more efficient algorithm than exhaustive search through all possible configurations.

The rules of an ***n-player game*** G are set by families f, v of ***information*** and ***value*** functions and a ***transition rule*** r. Each player $i \in I$ at each step participates in transforming a configuration (game position) $x \in C$ into the new configuration $r(x, m),\ m : I \to M$ by choosing a move $m_i = m(i)$ based only on his knowledge $f_i(x)$ of x. The game proceeds until a ***terminal*** configurations $t \in T \subset C$ is reached. Then $v_i(t)$ is the loss or gain of the i-th player. Our games will have ***zero sum*** $\sum v_i(t) = 0$ and ***full information:*** $f_i(x) = x$, $r(x, m) = r'(x, m_{a(x)})$, where $a(x)$ points to the ***active*** player. We consider binary, two-players, no-draw games, taking $0 \notin C \subset \mathbb{Z}$, $M \subset \mathbb{Z}$, $T = I = \{\pm 1\}$, $a(x) = \mathbf{sign}(x)$, $v_i(t) = a(t)i$, and $|r(x, m)| < |x|$.[5]

An example of such games is chess. Examples of games without full information are card games, where only a part $f_i(x)$ (player's own hand) of the position x is known. Each player may have a ***strategy*** providing a move for each position. A strategy S is ***winning*** at x if starting at a position x it guarantees victory whatever the opponent does, even if he knows S. We can extend v_1 on T to V on all positions with a winning strategy for one side so that $a(x)V(x) = \sup_m\{a(x)V(r(x, m))\}$. ($\sup\{\}$ taken as -1.)

Evaluating or ***solving*** a game, means computing V. This ability is close to the ability to find a good move in a modified game. Indeed, modify a game G into G' by adding a preliminary stage to it.
At this stage the player A offers a starting position for G and her opponent B chooses which side to play. Then A may either start playing G or decrement the counter of unused positions and offer another one. Obviously, B wins if he can determine the winning side of every position. If he cannot while A can, A wins. Also, any game can be modified into one with two moves: $M \subset \{0, 1\}$ by breaking a string move into several bit-moves. (A position of the new game consists of a position x of the old one and a prefix y of a move. The active player keeps adding bits to y until m is complete and the next position generated by $r(x, m)$.) Evaluating such games is obviously sufficient for choosing the right move.

Theorem. *Each position of any full information game has a winning strategy for one side.*

(This theorem [Neumann, Morgenstern 44] fails for games with ***partial*** information: either player may lose if his strategy is known to the adversary. E.g.: 1. Blackjack (21); 2. Each player picks a bit; their equality determines the winner.) The game can be solved by playing all strategies against each other. There are 2^n positions of length n, $(2^n)^{2^n} = 2^{n \times 2^n}$ strategies and $2^{n \times 2^{n+1}}$ pairs of them. For a 5-bit game that is 2^{320}. The proof of this Theorem gives a much faster (but still exponential time!) strategy.

Proof. Make the graph of all $\leq \|x\|$ -bit positions and moves; Set $V = 0$; reset $V = v$ on T.
Repeat until idle: If $V(x) = 0$, set $V(x) = a(x) \sup_m\{a(x)V(r(x, m))\}$.
The procedure stops with empty $V^{-1}(0)$ since $|r(x, m)| < |x|$ in our games keep decreasing. □

Games may be categorized by the difficulty to compute r. We will consider only r computable in linear space $O(\|x\|)$. Then, the $2^{2\|x\|}$ possible moves can be computed in exponential time, say $2^{3\|x\|}$. The algorithm tries each move in each step. Thus, its total running time is $2^{3\|x\|+1}$: *extremely* slow (2^{313} for a 13-byte game) but still *much* faster than the previous (double exponential) algorithm.

Exercise: the Match Game. Consider 3 boxes with 3 matches each: ! ! ! | ! ! ! | ! ! !.
The players alternate turns taking any *positive* number of matches from a *single* box. One cannot leave the table empty. Use the above algorithm to evaluate all positions and list the evaluations after each its cycle.

Exercise: Modify the chess game by giving one side the right to make (if it chooses to) an extra move out of turn during the first 10 moves. Prove that this side have a non-loosing strategy.

[5] Our examples will assure "$< |x|$" by implicitly prepending non-terminal configurations with a counter of remaining steps.

3.2 Exponentially Hard Games

A simple example of a full information game is ***Linear Chess***, played on a finite linear board. Each piece has a 1-byte type, including *loyalty* to one of two sides: W (weak) or S (shy), *gender* M/F and a 6-bit *rank*. All cells of the board are filled and all W's are always on the left of all S's. Changes occur only at the ***active*** border where W and S meet (and fight). The winner of a fight is determined by the following Gender Rules:

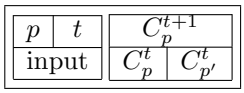
1. If S and W are of the same sex, W (being weaker) loses.
2. If S and W are of different sexes, S gets confused and loses.

The party of a winning piece A replaces the loser's piece B by its own piece C. The choice of C is restricted by the table of rules listing all allowed triples (ABC). We will see that this game *cannot* be solved in a *subexponential* time. We first prove that (see [Chandra, Kozen, Stockmeyer 81]) for an artificial game. Then we reduce this ***Halting Game*** to Linear Chess showing that any fast algorithm to solve Linear Chess, could be used to solve Halting Game, thus requiring exponential time. For Exp-Time Completeness of regular (but $n \times n$) Chess, Go, Checkers see: [Fraenkel, Lichtenstein 81, Robson 83, 84].

Exptime Complete Halting Game

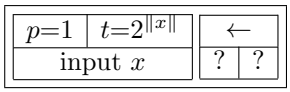
We use a universal Turing Machine u (defined as 1-pointer cellular automata) which halts only by its head rolling off of the tape's left end, leaving a blank. Bounded Halting Problem BHP(x) determines if $u(x)$ stops (i.e. the leftmost tape cell points left: $d{=}{-}1$) within $2^{\|x\|}$ steps. This cannot be determined in $o(2^{\|x\|})$ steps.

We now convert BHP into the Halting Game.
The players are: W claiming $u(x)$ halts in time (and should have winning strategy iff this is true); His opponent is S. The ***board*** has four parts: the C diagram, the input x to u, positive integers p (position) and t (time in the execution of $u(x)$):



The diagram shows the states C_p^{t+1}, C_p^t of cell p at times $t{+}1, t$, and $C_{p'}^t$ of cell $p' = p{+}d'$, $d' {\in} \{\pm 1\}$ at time t. C include present d and previous d' pointers direction; C^t may be replaced by "?". Some board configurations are illegal: if (1) both $C_p, C_{p'}$ point away from each other, or (2) C^{t+1} differs from the result prescribed by the transition rules for C^t, or (3) $t{=}1$, while $C_p^1 \neq x_p$. (At $t{=}1$, $u(x)$ is just starting, so its tape has the input x at the left starting with the head in the initial state, followed by blanks at the right.)

Here are the **Game Rules:** The game starts in the configuration at the right. W, in its moves, replaces the ?s with symbols claiming to reflect the state of cells p', p at step t of $u(x)$. S then chooses $s {\in} \{0, 1\}$, replaces p with $p{+}sd'$, moves C_p to top C box, fills lower C boxes with ?s, and decrements t:



Note that W may lie (i.e fill in "?" distorting the actual computation of $u(x)$), as long as he is consistent with the above "local" rules. All S can do is to check the two consecutive board configurations. He cannot refer to past moves or to actual computation of $u(x)$ as an evidence of W's violation.

Strategy: If $u(x)$ does indeed halt within $2^{\|x\|}$ steps, then the initial configuration is true to the computation of $u(x)$. Then W has an obvious (though hard to compute) winning strategy: just tell truly (and thus always consistently) what actually happens in the computation. S will lose when $t{=}1$ and cannot decrease any more. If the initial configuration is a lie, S can force W to lie all the way down to $t = 1$. How?

If the upper box C_p^{t+1} of a legal configuration is false then the lower boxes $C_{p'}^t$, C_p^t cannot both be true, since the rules of u determine C_p^{t+1} uniquely from them. If S correctly points the false C and brings it to the top on his move, then W is forced to keep on lying. At time $t{=}1$ the lie is exposed: the configuration doesn't match the actual input string x, i.e. is illegal.

Solving this game amounts to deciding correctness of the initial configuration, i.e. $u(x)$ halting in $2^{\|x\|}$ steps: impossible in time $o(2^{\|x\|})$. This Halting Game is artificial, still has a BHP flavor, though it does not refer to exponents. We now reduce it to a nicer game (Linear Chess) to prove it exponentially hard, too.

3.3 Reductions; Non-Deterministic and Alternating TM; Time and Space

To ***reduce*** (see definition in sec. 2.2) Halting game to Linear Chess we introduce a few concepts.

A ***non-deterministic*** Turing Machine (NTM) is a TM that sometimes offers a (restricted) transition choice, made by a ***driver***, a function (of the TM configuration) of unrestricted complexity. A deterministic (ordinary) TM M accepts a string x if $M(x)$=yes; an NTM M does if there exists a driver d s.t. $M_d(x)$=yes. NTM represent single player games – puzzles – with a simple transition rule, e.g., Rubik's Cube.
One can compute the winning strategy in exponential time by exhaustive search of all d.

Home Work: Prove all such games have P-time winning strategies, or show some have not.
Will get you grade A for the course, $1,000,000 Award and a senior faculty rank at a school of your choice.

The ***alternating*** TM (ATM) is a variation of the NTM driven by two alternating drivers (players) l, r. A string is accepted if there is l such that for any $r : M_{l,r}(x)$= yes. Our games could be viewed as ATM returning the result of the game in linear space but possibly exponential time, M prompts l and r alternatingly to choose their moves (in several steps for multi-bit moves) and computes the resulting position, until a winner emerges. Accepted strings describe winning (i.e. having a winning strategy) positions.

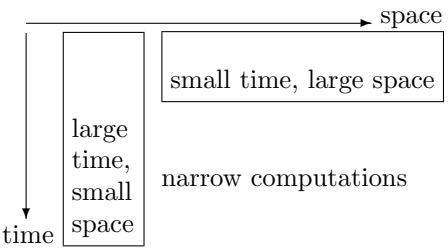
Linear Chess (LC) Simulation of TM-Games. The simulation first represents the Halting Game as an ATM computation simulated by the Ikeno TM (2.1) (using the "A/B" command for players' input). The UTM is viewed as an array of 1-pointer cellular automata: Weak cells as rightward, Shy leftward. Upon termination, the TM head is set to move to the end of the tape, eliminating all loser pieces.

This is viewed as a game of ***1d-Chess (1dC)***, a variant of LC, where the table, not the "Gender Rule", determine the victorious piece, and not only the vanquished piece is replaced, but also the winning piece may be "promoted to" another type of the same side. The types are states of Ikeno TM showing Loyalty (pointer direction) $d \in \{W, S\}$, gender g (= previous d), and 6/6/6/5 ranks (trit $t \in \{0, 1, *\}$ with $'$ bit p).

Exercise: Describe LC simulation of 1dC. **Hint:** Each 1dC transition is simulated in several LC stages. Let L,R be the two pieces active in 1dC. In odd stages L (in even stages R) changes gender while turning pointer twice. The last stage turns pointer only once and possibly changes gender. In the first stage L appends its rank with R's p bit. All other stages replace old 1dC rank with the new one. R appends its old t bit (only if $t \neq *$) to its new rank. Subsequent stages drop both old bits, marking L instead if it is the new 1dC head. Up to 4 more stages are used to exit any mismatch with 1dC new d, g bits.

Space-Time Trade-off. ***Deterministic*** linear space computations are games where any position has at most one (and easily computable) move. We know no general superlinear lower bound or subexponential upper bound for time required to determine their outcome. This is a big open problem.

Recall that on a parallel machine: ***time*** is the number of steps until the last processor halts; ***space*** is the amount of memory used; ***volume*** is the combined number of steps of all processors. "***Small***" will refer to values bounded by a polynomial of the input length; "***large***" to exponential. Let us call computations ***narrow*** if *either* time *or* space are polynomial, and ***compact*** if both (and, thus, volume too) are. **An open question:** Do all exponential volume algorithms (e.g., one solving Linear Chess) allow an equivalent *narrow* computation?



Alternatively, can every *narrow* computation be converted into a compact one? This is equivalent to the existence of a P-time algorithm for solving any ***fast*** game, i.e. a game with a P-time transition rule and a move counter limiting *the number of moves* to a polynomial. The sec. 3.1 algorithm can be implemented in parallel P-time for such games. Converse also holds, similarly to the Halting Game.

[Stockmeyer, Meyer 73] solve compact games in P-space: With $M \subset \{0, 1\}$ run depth-first search on the tree of all games – sequences of moves. On exiting each node it is marked as the active player's win if some move leads to a child so marked; else as his opponent's. Children's marks are then erased. Conversely, compact games can simulate any P-space algorithms. Player A declares the result of the space-k, time-2^k computation. If he lies, player B asks him to declare the memory state in the middle of that time interval, and so by a k-step binary search catches A's lie on a mismatch of states at two adjacent times. This has some flavor of trade-offs such as saving time at the expense of space in dynamic programming.

Thus, fast games (i.e. compact alternating computations) correspond to narrow deterministic computations; general games (i.e. narrow alternating computations) correspond to large deterministic ones.

Part II
Mysteries

We now enter Terra Incognita by extending deterministic computations with tools like random choices, nondeterministic guesses, etc., the power of which is completely unknown. Yet many fascinating discoveries were made there in which we will proceed to take a glimpse.

4 Nondeterminism; Inverting Functions; Reductions

4.1 An Example of a Narrow Computation: Inverting a Function

Consider a P-time function F. For convenience, assume $\|F(x)\| = \|x\|$, (often $\|F(x)\| = \|x\|^{\Theta(1)}$ suffices). Inverting F means finding, for a given y, at least one $x \in F^{-1}(y)$, i.e. such that $F(x) = y$.

We may try all possible x for $F(x) = y$. Assume F runs in linear time on a Pointer Machine. What is the cost of inverting F? The *space* used is $\|x\| + \|y\| + \text{space}_F(x) = O(\|x\|)$. But time is $O(\|x\|2^{\|x\|})$: absolutely infeasible. No method is currently proven much better in the worst case. And neither could we prove some inversion problems to require *super-linear* time. This is the sad present state of Computer Science!

An Example: Factoring. Let $F(x_1, x_2) = x_1 x_2$ be the product of integers. For simplicity, assume x_1, x_2 are primes. A fast algorithm in sec. 5.1 determines if an integer is prime. If not, no factor is given, only its existence. To invert F means to factor $F(x)$. The density of n-bit primes is $\approx 1/(n \ln 2)$. So, factoring by exhaustive search takes *exponential* time! In fact, even the best known algorithms for this ancient problem run in time about $2^{\sqrt{\|y\|}}$, despite centuries of efforts by most brilliant people. The task is now commonly believed infeasible and the security of many famous cryptographic schemes depends on this *unproven* faith.

One-Way Functions: $F : x \to y$ are those easy to compute ($x \mapsto y$) and hard to invert ($y \mapsto x$) for most x. Even their existence is sort of a religious belief in Computer Theory. It is unproven, though many functions *seem* to be one-way. Some functions, however, are proven to be one-way, IFF one-way functions EXIST. Many theories and applications are based on this hypothetical existence.

Search and NP Problems

Let us compare the inversion problems with another type – the search problems specified by computable in time $\|x\|^{O(1)}$ relations $F(x, w)$: given x, find w s.t. $F(x, w)$. There are two parts to a search problem: (a) decision problem: decide if w (called ***witness***) exist, and (b) a constructive problem: actually find w.

Any inversion problem is a search problem and any search problem can be restated as an inversion problem. E.g., finding a Hamiltonian cycle C in a graph G, can be stated as inverting a $f(G, C)$, which outputs $G, 0 \ldots 0$ if C is in fact a Hamiltonian cycle of G. Otherwise, $f(G, C) = 0 \ldots 0$.

Similarly any search problem can be reduced to another one equivalent to its decision version. For instance, factoring x reduces to bounded factoring: given x, b find p, q such that $pq = x$, $p \le b$ (where decisions yield construction by binary search).

Exercise: Generalize the two above examples to reduce any search problem to an inverting problem and to a decision problem.

The ***language*** of a problem is the set of all acceptable inputs. For an inversion problem it is the range of f. For a search problem it is the set of all x s.t. $F(x, w)$ holds for some w. An ***NP language*** is the set of all inputs acceptable by a P-time ***non-deterministic*** Turing Machine (sec. 3.3). All three classes of languages – search, inversion and NP – coincide (NP $\iff$ search is straightforward).

Interestingly, polynomial *space* bounded deterministic and non-deterministic TMs have equivalent power. It is easy to modify TM to have a unique accepting configuration. Any acceptable string will be accepted in time 2^s, where s is the space bound. Then we need to check $A(x, w, s, k)$: whether the TM can be driven from the configuration x to w in time $< 2^k$ and space s. For this we need for every z, to check $A(x, z, s, k-1)$ and $A(z, w, s, k-1)$, which takes space $t_k \le t_{k-1} + \|z\| + O(1)$. So, $t_k = O(sk) = O(s^2)$ [Savitch 70].

Search problems are games with P-time transition rules and one move duration. A great hierarchy of problems results from allowing more moves and/or other complexity bounds for transition rules.

4.2 Complexity of NP Problems

We discussed the (equivalent) inversion, search, and NP types of problems. Nobody knows whether *all* such problems are solvable in P-time (i.e. belong to P). This question (called P=?NP) is probably the most famous one in Theoretical Computer Science. All such problems are solvable in exponential time but it is unknown whether any better algorithms generally exist. For many problems the task of finding an efficient algorithm may seem hopeless, while similar or slightly modified problems have been solved. Examples:

1. Linear Programming: Given integer $n \times m$ matrix A and vector b, find a rational vector x with $Ax < b$.
   Note, if n and entries in A have $\leq k$-bits and x exists then an $O(nk)$-bit x exists, too.

   Solution: The Dantzig's ***Simplex*** algorithm finds x quickly for many A.
   Some A, however, take exponential time. After long frustrating efforts, a worst case
   P-time Ellipsoid Algorithm was finally found in [Yudin and A.S. Nemirovsky 76].
2. Primality test: Determine whether a given integer p has a factor?

   Solution: A bad (exponential time) way is to try all $2^{\|p\|}$ possible integer factors of p.
   More sophisticated algorithms, however, run fast (see section 5.1).
3. Graph Isomorphism Problem: Are two given graphs G_1, G_2, isomorphic?
   I.e., can the vertices of G_1 be re-numbered so that it becomes equal G_2?

   Solution: Checking all $n!$ enumerations of vertices is impractical
   (for $n = 100$, this exceeds the number of atoms in the known Universe).
   [Luks 80] found an $O(n^d)$ steps algorithm where d is the degree. This is a P-time for $d = O(1)$.
4. Independent Edges (Matching):
   Find a given number of independent (i.e., not sharing nodes) edges in a given graph.

   Solution: Max flow algorithm solves a bipartite graph case.
   The general case is solved with a more sophisticated algorithm by J. Edmonds.

Many other problems have been battled for decades or centuries and no P-time solution has been found. Even modifications of the previous four examples have no known answers:

1. Linear Programming: All known solutions produce rational x.
   No reasonable algorithm is known to find integer x.
2. Factoring: Given an integer, find a factor. Can be done in about exponential time $n^{\sqrt{n}}$.
   Seems very hard: Centuries of quest for fast algorithm were unsuccessful.
3. Sub-graph isomorphism: In a more general case of finding isomorphisms of a graph
   to a part of another, no P-time solution has been found, even for $O(1)$-degree graphs.
4. Independent Nodes: Find k independent (i.e., not sharing edges) nodes in a given graph.
   No P-time solution is known.

Exercise: Restate the above problems as inverting easily computable functions.

We learned the proofs that Linear Chess and some other games have exponential complexity. None of the above or any other search/inversion/NP problem, however, have been proven to require super-P-time. When, therefore, do we stop looking for an efficient solution?

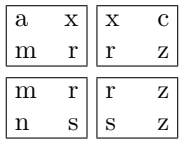
NP-Completeness theory is an attempt to answer this question.
See results by S.Cook, R.Karp, L.Levin, and others surveyed in [Garey, Johnson 79, Trakhtenbrot 84].

A P-time function f reduces one NP-predicate $p_1(x)$ to $p_2(x)$ iff $p_1(x) = p_2(f(x))$, for all x. p_2 is NP-complete if *all* NP problems can be reduced to it. Thus, each NP-complete problem is at least as worst-case hard as all other NP problems. This may be a good reason to give up on fast algorithms for it. Any P-time algorithm for one NP-complete problem would yield one for all other NP (or inversion, or search) problems. No such solution has been discovered yet and this is left as a homework (10 years deadline).

Faced with an NP-complete problem we can sometimes restate it, find a similar one which is easier (possibly with additional tools) but still gives the information we really want. We will do this in Sec. 5.1 for factoring. Now we proceed with an example of NP-completeness.

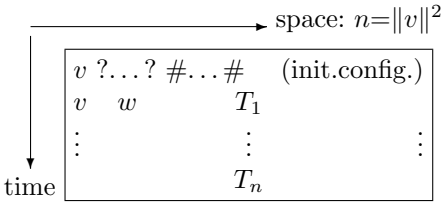
4.3 An NP-Complete Problem: Tiling

Tiling Problem. Invert the function which, given a tiled square, outputs its first row and the list of tiles used. A tile is one of the 26^4 possible squares containing a Latin letter at each corner. Two tiles may be placed next to each other if the letters on the shared side match. (See an example at the right.) We now reduce to Tiling any search problem: given x, find w satisfying a P-time computable property $F(x, w)$.



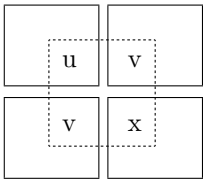
Padding Argument. First, we need to reduce it to some "standard" NP problem. An obvious candidate is the problem "Is there $w : U(v, w)$?", where U is the universal Turing Machine, simulating $F(x, w)$ for $v = px$. But U does not run in P-time, so we must restrict U to u which stops within some P-time limit. How to make this fixed degree limit sufficient to simulate any polynomial (even of higher degree) time? Let the TM $u(v, w)$ for v=$00 \dots 01px$ simulate $\|v\|^2$ steps of $U(px, w)$=$F(x, w)$. If the ***padding*** of 0's in v is sufficiently long, u will have enough time to simulate F, even though u runs in quadratic time, while F's time limit may be, say, cube (of a shorter "un-padded" string). So the NP problem $F(x, w)$ is reduced to $u(v, w)$ by mapping instances x into $f(x) = 0 \dots 01px = v$, with $\|v\|$ determined by the time limit for F. Notice that program p for F is fixed. So, if some NP problem *cannot* be solved in P-time then neither can be the problem $\exists ?w : u(v, w)$. Equivalently, if the latter *IS* solvable in P-time then so is *any* search problem. We do not know which of these alternatives is true. It remains to reduce the search problem u to Tiling.

The Reduction. We compute $u(v, w)$ (where $v = 00 \dots 01px$) by a TM represented as an array of 1-pointer cellular automata that runs for $\|v\|^2$ steps and stops if w does NOT solve the relation F. Otherwise it enters an infinite loop. An instance x has a solution iff $u(v, w)$ runs forever for some w and $v = 0 \dots 01px$. Here is the space-time diagram of computation of $u(v, w)$. We set n to u's time (and space) $\|v\|^2$. Each row in this table represents the configuration of u in the respective moment of time. The solution w is filled in at the second step below a special symbol "?". If a table is filled in wrongly, i.e. doesn't reflect any actual computation, then it must have four cells sharing a corner that couldn't possibly appear in the computation of u on any input.



Proof. As the input v and the guessed solution w are the same in both the right and the wrong tables, the first 2 rows agree. The actual computation starts on the third row. Obviously, in the first mismatching row a transition of some cell from the previous row is wrong. This is visible from the state in both rows of this cell and the cell it points to, resulting in an impossible combination of four cells sharing a corner.

For a given x, the existence of w satisfying $F(x, w)$ is equivalent to the existence of a table with the prescribed first row, no halting state, and permissible patterns of each four adjacent squares (cells). Converting the table into the ***Tiling Problem***: The cells in the table are separated by "—" ; the tiles by "..."; Cut each cell into 4 parts by a vertical and a horizontal lines through its center and copy cell's content in each part. Combine into a tile each four parts sharing a corner of 4 cells. If these cells are permissible in the table, then so is the respective tile.



So, any P-time algorithm extending a given first row to the whole table of matching tiles from a given set could be used to solve any NP problem by converting it to Tiling as shown.

Exercise: Find a polynomial time algorithm for $n \times \log n$ Tiling Problem.

5 Probability in Computing

5.1 A Monte-Carlo Primality Tester

The factoring problem seems very hard. But to test if a number has factors turns out to be much easier than to find them. It also helps if we supply the computer with a coin-flipping device. See: [Miller 76, Solovay, Strassen 77, Rabin 80]. We now consider a Monte Carlo algorithm, i.e. one that with high probability rejects any composite number, but never a prime.

Residue Arithmetic. $p|x$ means p divides x. $x \equiv y \pmod p$ means $p|(x-y)$. $y = (x \bmod p)$ denotes the residue of x when divided by p, i.e. $x \equiv y \in [0, p-1]$. Residues can be added, multiplied and subtracted with the result put back in the range $[0, p-1]$ via shifting by an appropriate multiple of p. E.g., $-x$ means $p-x$ for residues mod p. We use $\pm x$ to mean either x or $-x$.

The Euclidean Algorithm finds $\gcd(x, y)$ – the greatest (and divisible by any other) common divisor of x and y: $\gcd(x, 0) = x$; $\gcd(x, y) = \gcd(y, (x \bmod y))$, for $y > 0$. By induction, $g = \gcd(x, y) = A * x - B * y$, where integers $A = (g/x \bmod y)$ and $B = (g/y \bmod x)$ are produced as a byproduct of that algorithm. This allows division (mod p) by any r ***coprime*** with p, (i.e. $\gcd(r, p) = 1$), and operations $+, -, *, /$ obey all usual arithmetical laws. We also need to compute $(x^q \bmod p)$ in polynomial time. We cannot do $q > 2^{\|q\|}$ multiplications. Instead we compute all numbers $x_i = (x_{i-1}^2 \bmod p) = (x^{2^i} \bmod p), i < \|q\|$. Then we represent q in binary, i.e. as a sum of powers of 2 and multiply mod p the needed x_i's.

Fermat Test. The Little Fermat Theorem for each prime $p \not| x$ says: $x^{(p-1)} \equiv 1 \pmod p$. Indeed, the sequence $(xi \bmod p)$ is a permutation of $[1, p-1]$. So, $1 \equiv (\prod_{i<p}(xi))/(p-1)! \equiv x^{p-1} (\bmod p)$. This test rejects typical composite p, including all $p = a^2 b \neq b : (1+p/a)^{p-1} = 1 + (p/a)(p-1) + (p/a)^2 c \equiv 1 - p/a \not\equiv 1 \pmod p$. Other composite p (***Carmichael numbers***) can be actually factored by the following tests.

Square Root Test. For each y and prime p, $x^2 \equiv y \pmod p$ has at most one pair of solutions $\pm x$.

Proof. Let x, x' be two solutions: $y \equiv x^2 \equiv x'^2 \pmod p$. Then $x^2 - x'^2 = (x-x')(x+x') \equiv 0 \pmod p$. So, $p \,|\, (x-x')(x+x')$. Thus p, if prime, divides either $(x-x')$ or $(x+x')$, making $x \equiv \pm x'$. Otherwise p is composite, and $\gcd(p, x+x')$ *actually gives* its factor.

Random Choice. We say d ***kills*** $\mathbb{Z}_p^*$ if $x^d \equiv 1 \pmod p$ for all x (in $\mathbb{Z}_p^*$, i.e. coprime with p). If $x^d \not\equiv 1$, then for all y either $y^d \not\equiv 1$ or $(xy)^d \not\equiv 1$. Same with $x^d \not\equiv \pm 1 \pmod p$. So existence of a single such x implies the same for **most** of randomly chosen y.

Miller-Rabin Test T_x factors a composite p given d that kills $\mathbb{Z}_p^*$. If $d = p-1$ does not, then Fermat Test confirms p is composite. Let $d = 2^k q$, with odd q. T_x sets $x_0 = (x^q \bmod p)$, $x_i = (x_{i-1}^2 \bmod p) = (x^{2^i q} \bmod p)$, $i \leq k$. $x_k = 1$. If $x_0 = 1$, or one of x_i is -1, T_x gives up for this x. Otherwise $x_i \not\equiv \pm 1$ for some $i < k$, while $x_i^2 \equiv x_{i+1} \equiv 1$, and the Square Root Test factors p.

Now, for any coprime a, b, $p = ab$, T succeeds with some (thus most!) $x \in \mathbb{Z}_p^*$: Take the **greatest** i such that $2^i q$ does not kill $\mathbb{Z}_p^*$. It exists (as $(-1)^q \equiv -1$ for odd q) and has $x_i \not\equiv 1 \equiv (x_i)^2 \pmod p$. Then $x' = 1 + b(1/b \bmod a)(x-1) \equiv 1 \equiv x_i' \pmod b$, while $x_i' \equiv x_i \not\equiv 1 \pmod a$. So, $x_i' \not\equiv \pm 1 \pmod p$.

5.2 Randomized Algorithms and Random Inputs

Las-Vegas algorithms, unlike Monte-Carlo, never give wrong answers. Unlucky coin-flips just make them run longer than expected. Quick-Sort is a simple example. It is about as fast as deterministic sorters, but is popular due to its simplicity. It sorts an array $a[1 \dots n]$ of $n > 2$ numbers by choosing in it a random ***pivot***, splitting the remaining array in two by comparing with the pivot, and calling itself recursively on each half.

For easy reference, rename the array entries with their positions $1, \dots, n$ in the *sorted output* (no effect on the algorithm). Denote $t(i)$ the (random) time i is chosen as a pivot. Then i will ever be compared with j iff either $t(i)$ or $t(j)$ is the smallest among $t(i), \dots, t(j)$. This has 2 out of $|j-i|+1$ chances. So, the expected number of comparisons is $\sum_{i,j>i} 2/(1+j-i) = -4n + (n+1)\sum_{k=1}^{n} 2/k = 2n(\ln n - O(1))$. Note, that the expectation of the sum of variables is the sum of their expectations (not true, say, for product).

The above Monte-Carlo and Las-Vegas algorithms require choosing strings *at random* with uniform distribution. We mentally picture that as flipping a coin. (Computers use ***pseudo-random generators*** rather than coins in hope, rarely supported by proofs, that their outputs have all the statistical properties of truly random coin flips needed for the analysis of the algorithm.)

Random Inputs to Deterministic Algorithms are analyzed similarly to algorithms that flip coins themselves and the two should not be confused. Consider an example: Someone is interested in knowing whether or not certain graphs contain Hamiltonian Cycles. He offers graphs and pays \$100 if we show either that the graph *has* or that it *has not* Hamiltonian Cycles. Hamiltonian Cycle problem is NP-Complete, so it should be very hard for *some*, but not necessarily for *most* graphs. In fact, if our patron chooses the graphs uniformly, a fast algorithm can earn us the \$100 *most of the time*! Let all graphs have n nodes and, say, $d < (\ln n)/2$ mean degree and be equally likely. Then we can use the following (deterministic) algorithm:
Output "**No** Hamiltonian Cycles" and collect the \$100, if the graph has an isolated node. Otherwise, pass on that graph and the money. Now, how often do we get our \$100. The probability that a given node A of the graph is isolated is $(1-1/n)^{dn} > (1-O(1/n))/\sqrt{n}$. Thus, the probability that *none* of n nodes is isolated (and we lose our \$100) is $O((1-1/\sqrt{n})^n) = O(1)/e^{\sqrt{n}}$ and vanishes fast. Similar calculations can be made whenever $r = \lim(d/\ln n) < 1$. If $r > 1$, other fast algorithms can actually find a Hamiltonian Cycle.
See: [Johnson 84, Karp 76, Gurevich 85]. See also [Levin Venkatesan 18] for a proof that another graph problem is NP-complete even on average. How do this HC algorithm and the above primality test differ?

- The primality algorithm works for *all* instances. It tosses the coin itself and can repeat it for a more reliable answer. The HC algorithms only work for *most* instances (with isolated nodes or generic HC).
- In the HC algorithms, we must trust the customer to follow the presumed random procedure. If he cheats and produces rare graphs often, the analysis breaks down.

Symmetry Breaking. Randomness comes into Computer Science in many other ways besides those we considered. Here is a simple example: avoiding conflicts for shared resources.

Dining Philosophers. They sit at a circular table. Between each pair is either a knife or a fork, alternating. The problem is, neighboring diners must share the utensils, cannot eat at the same time. How can the philosophers complete the dinner given that all of them must act in the same way without any central organizer? Trying to grab the knives and forks at once may turn them into fighting philosophers. Instead they could each flip a coin, and sit still if it comes up heads, otherwise try to grab the utensils. If two diners try to grab the same utensil, neither succeeds. If they repeat this procedure enough times, most likely each philosopher will eventually get both a knife and a fork without interference.

We have no time to actually analyze this and many other scenaria, where randomness is crucial. Instead we will take a look into the concept of Randomness itself.

5.3 Arithmetization: One-Player Games with Randomized Transition

In section 3 Games, to win, the player (call him Merlin), must beat another perfectly powerful wizard (say, Lady of the Lake). To make the challenge realistic we remove the Lady, giving Merlin to evaluate, rather than choose, moves. Its correctness is challenged by a randomized transition, performed by a simple player, Arthur. He chooses moves for which Merlin's assessment is almost surely wrong unless it is correct for all moves. The wrong assessment will not match the assessment after the next move, unless it is wrong, too, etc.

The trick achieving this, called ***arithmetization***, was proposed in Noam Nisan's article widely distributed over email in the Fall of 1989 and quickly used in a flood of follow-ups for proving relations between various complexity classes. We follow [Shamir 90, Fortnow, Lund 93]. The trick is based on the feature that degree d polynomials coincide on the whole field if they do on more than d points. To use it we express the boolean functions as low degree polynomials, and apply them to $\mathbb{Z}_p$-tokens (let us call them ***bytes***) instead of bits.

This reduces generic games to games in which any Merlin's strategy in any losing configuration has exponentially small chance to win. The reduction holds for games with any (exponential, polynomial, etc.) limit on the remaining moves counter c. This c will be included implicitly in games configurations below.

Let g be the (ATM-complete) game of 1d-Chess (3.3), $r(m,x)$ with $x{=}x_1 \dots x_s$, $m, x_i{\in}\{0,1\}$ be its transition rule. Configurations include x and a remaining moves counter $c \le 2^s$. They are terminal if $c{=}0$, winning to the player x_1. Intermediate configurations (m,x,y) have y claimed as a prefix of $r(m,x)$.

Let $t(m,x,y)$ be 1 if $y{=}r(m,x)$, else $t{=}0$. 1d-Chess is simple, so t can be expressed as a product of s multilinear $O(1)$-sized terms, any variable shared by at most two terms. Thus t is a polynomial, quadratic in each m, x_i, y_i. Let $V_c(x)$ be 1 if the active player has a strategy to win in the c moves left, i.e. $V_0(x){:=}\,x_1$, $V_{c+1}(x){:=}\ 1{-}V'_c(0,x,\{\})V'_c(1,x,\{\}) = 1{-}V_c(r(0,x))V_c(r(1,x))$, where $V'_c(m,x,y){:=}\,V_c(y)t(m,x,y)$ for $y = y_1 \dots y_s$ or $V'_c(m,x,y){:=}\,V'_c(m,x,y{\circ}0){+}V'_c(m,x,y{\circ}1)$ for shorter y. ($\circ$ stands for concatenation.)

G will allow Merlin to prove x is winning i.e., $V_c(x) = 1$. At the start Merlin chooses a $2s$-bit prime p. Configurations $X = (m,x,y,v)$ of G replace x_i, m, y_i bits with $\mathbb{Z}_p$-bytes and add $v{\in}\mathbb{Z}_p$ reflecting Merlin's claimed $v{=}V'_c(m,x,y)$. The polynomial V_c retains the above inductive definition, thus is quadratic in each x_i, m, as $t(m,x,y)$ is. Then y_i have degree ≤ 4 in $V_c(y)$ and ≤ 6 in $V_c(m,x,y)$.

At his steps Merlin chooses a univariate polynomial P of degree 6. v must be $1{-}P(1)P(0)$ for s-byte y, or $P(0){+}P(1)$ for shorter y. Arthur then chooses a random $r{\in}\mathbb{Z}_p$ and X becomes $(r, y, \{\}, P(r))$, or $(m, x, y{\circ}r, P(r))$, respectively. For X with a correct v Merlin's obvious winning strategy is to always provide the correct P. If v is wrong then either $P(1)$ or $P(0)$ must be wrong, too. So P will differ from V and they can agree only on few (bounded by degree) points. Thus $P(r)$ will be correct only on exponentially small fraction of random r. So, the wrong v will propagate throughout the game until it becomes obvious at $c{=}0$. This gives any Merlin strategy an exponentially small winning chance.

This reduction of Section 3 games yields a hierarchy of Arthur-Merlin games powers, i.e. the type of computations that have reductions to $V_c(x)$ of such games and back. The one-player games with randomized transition rule r running in space linear in the size of initial configuration are equivalent to exponential time deterministic computations. If instead the running time T of r combined for all steps is limited by a polynomial, then the games are equivalent to polynomial space deterministic computations.

An interesting twist comes in one move games with polylog T, too tiny to examine the initial configuration x and the Merlin's move m. But not only this obstacle is removed but the equivalence to NP is achieved with a little care. Namely, x is set in an error-correcting code, and r is given $O(\log \|x\|)$ coin-flips and random access to the digits of x, m. Then the membership proof m is reliably verified by the randomized r.
See [Holographic proof] for details and references.

6 Randomness

6.1 Randomness and Complexity

Intuitively, a random sequence is one that has the same properties as a sequence of coin flips. But this definition leaves the question, what *are* these properties? Kolmogorov resolved these problems with a new definition of random sequences: those with no description noticeably shorter than their full length. See survey and history in [Kolmogorov, V.A.Uspenskii 87, Li, Vitanyi 19].

Kolmogorov Complexity $K_A(x|y)$ of the string x given y is the length of the shortest program p which lets algorithm A transform y into x: $\min\{(\|p\|) : A(p,y) = x\}$. There exists a Universal Algorithm U such that, $K_U(x) \leq K_A(x) + O(1)$, for every algorithm A. This constant $O(1)$ is bounded by the length of the program U needs to simulate A. We abbreviate $K_U(x|y)$ as $K(x|y)$, or $K(x)$ for empty y.

An example: For $A : x \mapsto x$, $K_A(x) = \|x\|$, so $K(x) < K_A(x) + O(1) < \|x\| + O(1)$.

Can we compute $K(x)$ by trying all programs $p, \|p\|<\|x\|+O(1)$ to find the shortest one generating x? This does not work because some programs diverge, and the halting problem is unsolvable.
In fact, no algorithm can compute K or even any its lower bounds except $O(1)$.

Consider the Berry Paradox expressed in the phrase: "The smallest integer which cannot be uniquely and clearly defined by an English phrase of less than two hundred characters."
There are $< 128^{200}$ English phrases of < 200 characters. So there must be integers not expressible by such phrases and the smallest one among them. But isn't it described by the above phrase?

A similar argument proves that K is not computable. Suppose an algorithm $L(x) \neq O(1)$ computes a lower bound for $K(x)$. We can use it to compute $f(n)$ that finds x with $n < L(x) \leq K(x)$, but $K(x) < K_f(x)+O(1)$ and $K_f(f(n)) \leq \|n\|$, so $n < K(f(n)) < \|n\| + O(1) = \log O(n) \ll n$: a contradiction.
So, K and its non-constant lower bounds are not computable.

An important application of Kolmogorov Complexity measures the Mutual Information: $I(x : y) = K(x) + K(y) - K(x, y)$. It has many uses which we cannot consider here.

Deficiency of Randomness

Some upper bounds of $K(x)$ are close in some important cases. One such case is of x generated at random. Define its ***rarity*** for uniform on $\{0,1\}^n$ distribution as $d(x) = n - K(x|n) \geq -O(1)$.

What is the probability of $d(x) > i$, for uniformly random n-bit x ? There are 2^n strings x of length n. If $d(x) > i$, then $K(x|n) < n - i$. There are $< 2^{n-i}$ programs of such length, generating $< 2^{n-i}$ strings. So, the probability of such strings is $< 2^{n-i}/2^n = 2^{-i}$ (regardless of n)! Even for $n = 1,000,000$, the probability of $d(x) > 300$ is absolutely negligible (provided x was indeed generated by fair coin flips).

Small rarity implies all other enumerable properties of random strings. Indeed, let such property "$x \not\in P$" have a negligible probability and S_n be the number of n-bit strings violating P, so $s_n = \log(S_n)$ is small. To generate x, we need only the algorithm enumerating S_n and the s_n-bit position of x in that enumeration. Then the rarity $d(x) > n - (s_n+O(1))$ is large. Each x violating P will thus also violate the "small rarity" requirement. In particular, the small rarity implies unpredictability of bits of random strings: A short algorithm with high prediction rate would assure large $d(x)$. However, the randomness can only be refuted, cannot be confirmed: we saw, K and its lower bounds are not computable.

Rectification of Distributions. We rarely have a source of randomness with precisely known distribution. But there are very efficient ways to convert "roughly" random sources into perfect ones. Assume, we have such a sequence with weird unknown distribution. We only know that its long enough (m bits) segments have min-entropy $> k + i$, i.e. probability $< 1/2^{k+i}$, given all previous bits. (Without such m we would not know a segment needed to extract even one not fully predictable bit.) No relation is required between n, m, i, k, but useful are small m, i, k and huge $n = o(2^k/i)$. We can fold X into an $n \times m$ matrix. We also need a small $m \times i$ matrix Z, independent of X and **really** uniformly random (or random Toeplitz, i.e. with restriction $Z_{a+1,b+1} = Z_{a,b}$). Then the $n \times i$ product XZ has uniform with accuracy $O(\sqrt{ni/2^k})$ distribution. This follows from [Goldreich, Levin 89], which uses earlier ideas of U. and V. Vazirani.

6.2 Pseudo-randomness

The above definition of randomness is very robust, if not practical. True random generators are rarely used in computing. The problem is *not* that making a true random generator is impossible: we just saw efficient ways to perfect the distributions of biased random sources. The reason lies in many extra benefits provided by pseudorandom generators. E.g., when experimenting with, debugging, or using a program one often needs to repeat the exact same sequence. With a truly random generator, one actually has to record all its outcomes: long and costly. The alternative is to generate ***pseudo-random*** strings from a short seed. Such methods were justified in [Blum, Micali 84, Yao 82]:

First, take any one-way permutation $F_n(x)$ (see sec. 6.3) with a ***hard-core*** bit (see below) $B_p(x)$ which is easy to compute from x, p, but infeasible to guess from $p, n, F_n(x)$ with any noticeable correlation.
Then take a random ***seed*** of three k-bit parts x_0, p, n and Repeat: ($S_i \leftarrow B_p(x_i)$; $x_{i+1} \leftarrow F_n(x_i)$; $i \leftarrow i+1$).

We will see how distinguishing outputs S of this generator from strings of coin flips would imply the ability to invert F. This is infeasible if F is one-way. But if P=NP (a famous open problem), no one-way F, and no pseudorandom generators could exist.

By Kolmogorov's standards, pseudo-random strings are not random: let G be the generator; s be the seed, $G(s) = S$, and $\|S\| \gg k = \|s\|$. Then $K(S) \leq O(1) + k \ll \|S\|$, thus violating Kolmogorov's definition. We can distinguish between truly random and pseudo-random strings by simply trying all short seeds. However this takes time exponential in the seed length. Realistically, pseudo-random strings will be as good as a truly random ones if they can't be distinguished in feasible time. Such generators we call ***perfect***.

Theorem: [Yao 82] Let $G(s) = S \in \{0,1\}^n$ run in time t_G. Let a probabilistic algorithm A in expected (over internal coin flips) time t_A accept $G(s)$ and truly random strings with different by d probabilities. Then, for random i, one can use A to guess S_i from $S_{i+1}, S_{i+2}, \ldots$ in time $t_A + t_G$ with correlation $d/O(n)$.

Proof. Let r_i be the probability that A accepts $S = G(s)$ modified by replacing its first i digits with truly random bits. Then r_0 is the probability of accepting $G(s)$ and must differ by d from the probability r_n of accepting random string. Then $r_{i-1} - r_i = d/n$, for randomly chosen i.
Let R_0 and R_1 be the probabilities of accepting $r0x$ and $r1x$ for $x = S_{i+1}, S_{i+2}, \ldots$, and random $(i-1)$-bit r. Then $(R_1+R_0)/2$ averages to r_i, while $R_{S_i} = R_0+(R_1-R_0)S_i$ averages to r_{i-1} and $(R_1-R_0)(S_i-1/2)$ to $r_{i-1}-r_i = d/n$. So, R_1-R_0 has the stated correlation with S_i. □

If the above generator was not perfect, one could guess S_i from the sequence $S_{i+1}, S_{i+2}, \ldots$ with a polynomial (in $1/\|s\|$) correlation. But, $S_{i+1}, S_{i+2} \ldots$ can be produced from p, n, x_{i+1}. So, one could guess $B_p(x_i)$ from $p, n, F(x_i)$ with correlation d/n, which cannot be done for hard-core B.

Hard Core. The key to constructing a pseudorandom generator is finding a hard core for a one-way F. The following B is hard-core for any one-way F, e.g., for Rabin's OWF in sec. 6.3.
[Knuth 97] has more details and references.

Let $B_p(x) = (x \cdot p) = (\sum_i x_i p_i \bmod 2)$. [Goldreich, Levin 89] converts any method g of guessing $B_p(x)$ from $p, n, F(x)$ with correlation ε into an algorithm of finding x, i.e. inverting F (slower ε^2 times than g).

Proof. (Simplified with some ideas of Charles Rackoff.) Take $k = \|x\| = \|y\|$, $j = \log(2k/\varepsilon^2)$, $v_i = 0^i 10^{k-i}$. Let $B_p(x) = (x \cdot p)$ and $b(x,p) = (-1)^{B_p(x)}$. Assume, for $y = F_n(x)$, $g(y,p,w) \in \{\pm 1\}$ guesses $B_p(x)$ with correlation $\sum_p 2^{-\|p\|} b(x,p) g_p > \varepsilon$, where g_p abbreviates $g(y,p,w)$, since w, y are fixed throughout the proof.

$(-1)^{(x \cdot p)} g_p$ averaged over $>2k/\varepsilon^2$ random pairwise independent p deviates from its mean (over all p) by $<\varepsilon$ (and so is >0) with probability $> 1 - 1/2k$. The same for $(-1)^{(x \cdot [p+v_i])} g_{p+v_i} = (-1)^{(x \cdot p)} g_{p+v_i} (-1)^{x_i}$.

Take a random $k \times j$ binary matrix P. The vectors Pr, $r \in \{0,1\}^j \setminus \{0^j\}$ are pairwise independent. So, for a fraction $\geq 1 - 1/2k$ of P, $\text{sign}(\sum_r (-1)^{xPr} g_{Pr+v_i}) = (-1)^{x_i}$. We could thus find x_i for all i with probability $> 1/2$ if we knew $z = xP$. But z is short: we can try all its 2^j possible values and check $y = F_n(x)$ for each !

So the inverter, for a random P and all i, r, computes $G_i(r) = g_{Pr+v_i}$. It uses Fast Fourier on G_i to compute $h_i(z) = \sum_r b(z,r) G_i(r)$. The sign of $h_i(z)$ is the i-th bit for the z-th member of output list. □

6.3 Cryptography

Rabin's One-way Function. Pick random prime numbers $p, q, \|p\| = \|q\|$ with two last bits =1, i.e. with odd $(p-1)(q-1)/4$. Then $n = pq$ is called a Blum number. Its length should make factoring infeasible.

Let $Q_n = (\mathbb{Z}_n^*)^2$ be the set of squares, i.e. ***quadratic residues*** (all residues are assumed $\pmod n$).

Lemma. Let $n = pq$ be a Blum number, $F : x \mapsto x^2 \in Q_n$. Then (1) F is a permutation on Q_n and (2) The ability to invert F on random x is equivalent to that of factoring n.

Proof. (1) $t=(p-1)(q-1)/4$ is odd, so $u=(t+1)/2$ is an integer. Let $x=F(z)$. Both $p-1$ and $q-1$ divide $2t$. So, by Fermat's little theorem, both p, q (and, thus n) divide $x^t-1 \equiv z^{2t}-1$. Then $F(x)^u \equiv x^{2u} = xx^t \equiv x$.

(2) The above y^u inverts F. Conversely, let $F(A(y)) = y$ for a fraction ε of $y \in Q_n$.
Each $y \in Q_n$ has $x, x' \neq \pm x$ with $F(x)=F(x')=y$, both with equal chance to be chosen at random.
If $F(x)$ generates y while $A(y) = x'$ the Square Root Test (5.1) has both x, x' for factoring n. □

Such one-way permutations, called "trap-door", have many applications; we look at cryptography below.

Picking random primes is easy: they have density $1/O(\|p\|)$. Indeed, one can see that $\binom{2n}{n}$ is divisible by every prime $p \in [n, 2n]$ but by no prime $p \in [\frac{2}{3}n, n]$ or prime power $p^i > 2n$. So, $(\log \binom{2n}{n}) / \log n = 2n/\log n - O(1)$ is an upper bound on the number of primes in $[n, 2n]$ and a lower bound on that in $[1, 2n]$ (and in $[3n, 6n]$ as a simple calculation shows). And fast VLSI exist to multiply long numbers and check primality.

Public Key Encryption. A perfect way to encrypt a message m is to add it $\bmod 2$ bit by bit to a random string S of the same length k. The resulting encryption $m \oplus S$ has the same uniform probability distribution, no matter what m is. So it is useless for the adversary who wants to learn something about m, without knowing S. A disadvantage is that the communicating parties must share a secret S as large as all messages to be exchanged, combined. ***Public Key*** Cryptosystems use two keys. One key is needed to encrypt the messages and may be completely disclosed to the public. The ***decryption*** key must still be kept secret, but need not be sent to the encrypting party. The same keys may be used repeatedly for many messages.

Such cryptosystem can be obtained [Blum, Goldwasser 82] by replacing the above random S by pseudo-random $S_i = (s_i \cdot x)$; $s_{i+1} = (s_i^2 \pmod n)$. Here a Blum number $n = pq$ is chosen by the Decryptor and is public, but p, q are kept secret. The Encryptor chooses $x \in \mathbb{Z}_2^{\|n\|}$, $s_0 \in \mathbb{Z}_n$ at random and sends $x, s_k, m \oplus S$. Assuming factoring is intractable for the adversary, S should be indistinguishable from random strings (even with known x, s_k). Then this scheme is as secure as if S were random. The Decryptor knows p, q and can compute u, t (see above) and $v = (u^{k-1} \bmod t)$. So, he can find $s_1 = (s_k^v \bmod n)$, and then S and m.

Another use of the intractability of factoring is digital signatures [Rivest, Shamir, Adleman 78, Rabin 79]. Strings x can be released as authorizations of $y = (x^2 \bmod n)$. Verifying x, is easy but the ability of forging it for generic y is equivalent to that of factoring n.

Go On!

You noticed that most of our burning questions are still open. Take them on!

Start with reading recent results (FOCS/STOC is a good source). See where you can improve them.
Start writing, first notes just for your friends, then the real papers. Here is a little writing advice:

A well written paper has clear components: skeleton, muscles, etc.
The skeleton is an acyclic digraph of basic definitions and statements, with cross-references.
The meat consists of proofs (muscles) each *separately* verifiable by competent graduate students having to read no other parts but statements and definitions cited. Intuitive comments, examples and other comfort items are fat and skin: a lack or excess will not make the paper pretty. Proper scholarly references constitute clothing, no paper should ever appear in public without! Trains of thought which led to the discovery are blood and guts: keep them hidden. Metaphors for other vital parts, like open problems, I skip out of modesty.

Writing Contributions. Section 1 was originally prepared by Elena Temin, Yong Gao and Imre Kifor (BU), others by Berkeley students: 2.3 by Mark Sullivan, 3.1 by Eric Herrmann and Elena Eliashberg, 3.2 by Wayne Fenton and Peter Van Roy, 3.3 by Carl Ludewig, Sean Flynn, and Francois Dumas, 4.1 by Jeff Makaiwi, Brian Jones and Carl Ludewig, 4.2 by David Leech and Peter Van Roy, 4.3 by Johnny and Siu-Ling Chan, 5.2 by Deborah Kordon, 6.1 by Carl Ludewig, 6.2 by Sean Flynn, Francois Dumas, Eric Herrmann, 6.3 by Brian Jones.

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