

CHAPTER 1

Set Theory

1.1 INTRODUCTION

This chapter treats some of the elementary ideas and concepts of set theory which are necessary for a modern introduction to probability theory.

1.2 SETS AND ELEMENTS, SUBSETS

A *set* may be viewed as any well-defined collection of objects, and they are called the *elements* or *members* of the set. We usually use capital letters, A, B, X, Y, \dots to denote sets, and lowercase letters, a, b, x, y, \dots to denote elements of sets. Synonyms for set are *class*, *collection*, and *family*.

The statement that an element a belongs to a set S is written

$$a \in S$$

(Here \in is the symbol meaning "is an element of".) We also write

$$a, b \in S$$

when both a and b belong to S .

Suppose every element of a set A also belongs to a set B , that is, suppose $a \in A$ implies $a \in B$. Then A is called a *subset* of B , or A is said to be *contained* in B , which is written as

$$A \subseteq B \text{ or } B \supseteq A$$

Two sets are equal if they both have the same elements or, equivalently, if each is contained in the other. That is,

$$A = B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A$$

The negations of $a \in A$, $A \subseteq B$, and $A = B$ are written $a \notin A$, $A \not\subseteq B$, and $A \neq B$, respectively.

Remark 1: It is common practice in mathematics to put a vertical line "|" or slanted line "/" through a symbol to indicate the opposite or negative meaning of the symbol.

Remark 2: The statement $A \subseteq B$ does not exclude the possibility that $A = B$. In fact, for any set A , we have $A \subseteq A$ since, trivially, every element in A belongs to A . However, if $A \subseteq B$ and $A \neq B$, then we say that A is a *proper subset* of A (sometimes written $A \subset B$).

Remark 3: Suppose every element of a set A belongs to a set B , and every element of B belongs to a set C . Then clearly every element of A belongs to C . In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The above remarks yield the following theorem.

Theorem 1.1: Let A, B, C be any sets. Then:

- (i) $A \subseteq A$.
- (ii) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- (iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Specifying Sets

There are essentially two ways to specify a particular set. One way, if possible, is to list its elements. For example,

$$A = \{1, 3, 5, 7, 9\}$$

means A is the set consisting of the numbers 1, 3, 5, 7, and 9. Note that the elements of the set are separated by commas and enclosed in braces $\{ \}$. This is called the *tabular form* or *roster method* of a set.

The second way, called the *set-builder form* or *property method*, is to state those properties which characterize the elements in the set, that is, properties held by the members of the set but not by nonmembers. Consider, for example, the expression

$$B = \{x : x \text{ is an even integer, } x > 0\}$$

which is read:

“ B is the set of x such that x is an even integer and $x > 0$ ”

It denotes the set B whose elements are positive even integers. A letter, usually x , is used to denote a typical member of the set; the colon is read as “such that” and the comma as “and.”

EXAMPLE 1.1

- (a) The above set A can also be written as

$$A = \{x : x \text{ is an odd positive integer, } x < 10\}$$

We cannot list all the elements of the above set B , but we frequently specify the set by writing

$$B = \{2, 4, 6, \dots\}$$

where we assume everyone knows what we mean. Observe that $9 \in A$ but $9 \notin B$. Also $6 \in B$, but $6 \notin A$.

- (b) Consider the sets

$$A = \{1, 3, 5, 7, 9\}, \quad B = \{1, 2, 3, 4, 5\}, \quad C = \{3, 5\}$$

Then $C \subseteq A$ and $C \subseteq B$ since 3 and 5, the elements C , are also members of A and B . On the other hand, $A \not\subseteq B$ since $7 \in A$ but $7 \notin B$, and $B \not\subseteq A$ since $2 \in B$ but $2 \notin A$.

- (c) Suppose a die is tossed. The possible “number” or “points” which appears on the uppermost face of the die belongs to the set $\{1, 2, 3, 4, 5, 6\}$. Now suppose a die is tossed and an even number appears. Then the outcome is a member of the set $\{2, 4, 6\}$ which is a (proper) subset of the set $\{1, 2, 3, 4, 5, 6\}$ of all possible outcomes.

Special Symbols, Real Line \mathbf{R} , Intervals

Some sets occur very often in mathematics, and so we use special symbols for them. Some such symbols follow:

\mathbf{N} = the *natural numbers* or positive integers:

$$\{1, 2, 3, \dots\}$$

\mathbf{Z} = all integers, positive, negative, and zero:

$$\{\dots, -2, -1, 0, 1, 2, \dots\}$$

\mathbf{R} = the real numbers

Thus we have $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{R}$.

The set \mathbf{R} of real numbers plays an important role in probability theory since such numbers are used for numerical data. We assume the reader is familiar with the graphical representation of \mathbf{R} as points on a straight line, as pictured in Fig. 1-1. We refer to such a line as the *real line* or the *real line \mathbf{R}* .

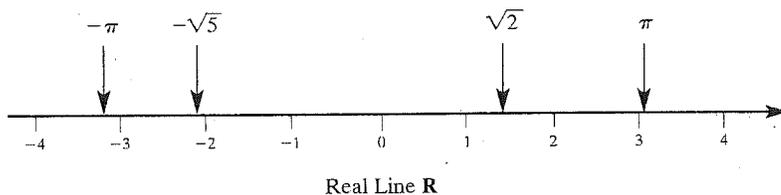


Fig. 1-1

Important subsets of \mathbf{R} are the *intervals* which are denoted and defined as follows (where a and b are real numbers with $a < b$):

Open interval from a to $b = (a, b) = \{x : a < x < b\}$

Closed interval from a to $b = [a, b] = \{x : a \leq x \leq b\}$

Open-closed interval from a to $b = (a, b] = \{x : a < x \leq b\}$

Closed-open interval from a to $b = [a, b) = \{x : a \leq x < b\}$

The numbers a and b are called the *endpoints* of the interval. The word “open” and a parenthesis “(” or “)” are used to indicate that an endpoint does not belong to the interval, whereas the word “closed” and a bracket “[” or “]” are used to indicate that an endpoint belongs to the interval.

Universal Set and Empty Set

All sets under investigation in any application of set theory are assumed to be contained in some large fixed set called the *universal set* or *universe of discourse*. For example, in plane geometry, the universal set consists of all the points in the plane; in human population studies, the universal set consists of all the people in the world. We will let

\mathbf{U}

denote the universal set unless otherwise stated or implied.

Given a universal set \mathbf{U} and a property P , there may be no elements in \mathbf{U} which have the property P . For example, the set

$$S = \{x : x \text{ is a positive integer, } x^2 = 3\}$$

has no elements since no positive integer has the required property. Such a set with no elements is called the *empty set* or *null set*, and is denoted by

$$\emptyset$$

There is only one empty set: If S and T are both empty, then $S = T$ since they have exactly the same elements, namely, none.

The empty set \emptyset is also regarded as a subset of every other set. Accordingly, we have the following simple result which we state formally:

Theorem 1.2: For any set A , we have $\emptyset \subseteq A \subseteq U$.

Disjoint Sets

Two sets A and B are said to be *disjoint* if they have no elements in common. Consider, for example, the sets

$$A = \{1, 2\}, \quad B = \{2, 4, 6\}, \quad C = \{4, 5, 6, 7\}$$

Observe that A and B are not disjoint since each contains the element 2, and B and C are not disjoint since each contains the element 4, among others. On the other hand, A and C are disjoint since they have no element in common. We note that if A and B are disjoint, then neither is a subset of the other (unless one is the empty set).

1.3 VENN DIAGRAMS

A *Venn diagram* is a pictorial representation of sets where sets are represented by enclosed areas in the plane. The universal set U is represented by the points in a rectangle, and the other sets are represented by disks lying within the rectangle. If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B , as in Fig. 1-2(a). If A and B are disjoint, that is, have no elements in common, then the disk representing A will be separated from the disk representing B , as in Fig. 1-2(b).

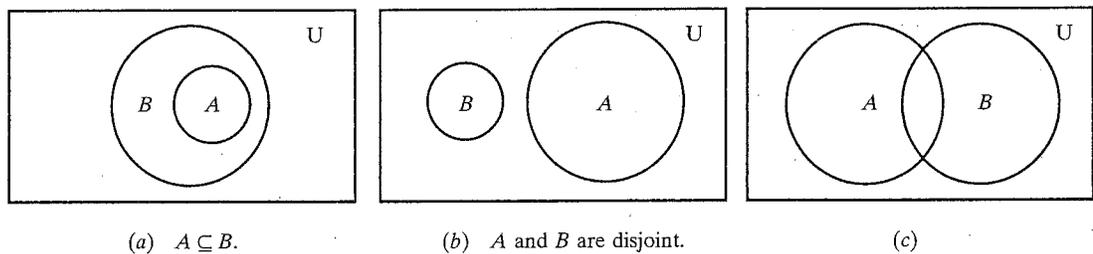


Fig. 1-2

On the other hand, if A and B are two arbitrary sets, it is possible that some elements are in A but not in B , some elements are in B but not in A , some are in both A and B , and some are in neither A nor B ; hence, in general, we represent A and B as in Fig. 1-2(c).

1.4 SET OPERATIONS

This section defines a number of set operations, including the basic operations of union, intersection, and complement.

Union and Intersection

The *union* of two sets A and B , denoted by $A \cup B$, is the set of all elements which belong to A or to B , that is,

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

Here, "or" is used in the sense of and/or. Figure 1-3(a) is a Venn diagram in which $A \cup B$ is shaded.

The *intersection* of two sets A and B , denoted by $A \cap B$, is the set of all elements which belong to both A and B , that is,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

Figure 1-3(b) is a Venn diagram in which $A \cap B$ is shaded.

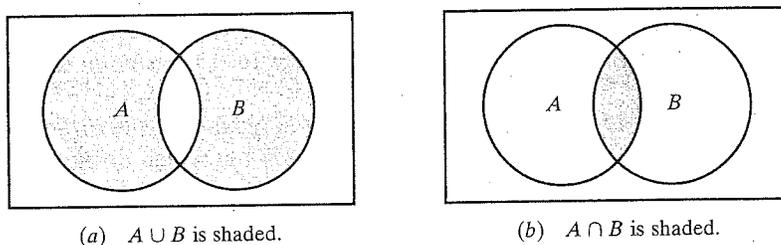


Fig. 1-3

Recall that sets A and B are said to be disjoint if they have no elements in common or, using the definition of intersection, if $A \cap B = \emptyset$, the empty set. If

$$S = A \cup B \text{ and } A \cap B = \emptyset$$

then S is called the *disjoint union* of A and B .

EXAMPLE 1.2

(a) Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6, 7\}$, $C = \{2, 3, 8, 9\}$. Then

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4, 5, 6, 7\}, & A \cup C &= \{1, 2, 3, 4, 8, 9\}, & B \cup C &= \{3, 4, 5, 6, 7, 8, 9\}, \\ A \cap B &= \{3, 4\}, & A \cap C &= \{2, 3\}, & B \cap C &= \{3\} \end{aligned}$$

(b) Let U be the set of students at a university, and let M and F denote, respectively, the sets of male and female students. Then U is the disjoint union of M and F , that is,

$$U = M \cup F \text{ and } M \cap F = \emptyset$$

This comes from the fact that every student in U is either in M or in F , and clearly no students belong to both M and F , that is, M and F are disjoint.

The following properties of the union and intersection should be noted:

(i) Every element x in $A \cap B$ belongs to both A and B ; hence, x belongs to A and x belongs to B . Thus, $A \cap B$ is a subset of A and of B , that is,

$$A \cap B \subseteq A \text{ and } A \cap B \subseteq B$$

(ii) An element x belongs to the union $A \cup B$ if x belongs to A or x belongs to B ; hence, every element in A belongs to $A \cup B$, and every element in B belongs to $A \cup B$. That is,

$$A \subseteq A \cup B \text{ and } B \subseteq A \cup B$$

We state the above results formally.

Theorem 1.3: For any sets A and B , we have

$$A \cap B \subseteq A \subseteq A \cup B \quad \text{and} \quad A \cap B \subseteq B \subseteq A \cup B$$

The operations of set inclusion is closely related to the operations of union and intersection, as shown by the following theorem (proved in Problem 1.16).

Theorem 1.4: The following are equivalent: $A \subseteq B$, $A \cap B = A$, $A \cup B = B$.

Other conditions equivalent to $A \subseteq B$ are given in Problem 1.55.

Complements, Difference, Symmetric Difference

Recall that all sets under consideration at a particular time are subsets of a fixed universal set U . The *absolute complement* or, simply, *complement* of a set A , denoted by A^c , is the set of elements which belong to U but which do not belong to A , that is,

$$A^c = \{x : x \in U, x \notin A\}$$

Some texts denote the complement of A by A' or \bar{A} . Figure 1-4(a) is a Venn diagram in which A^c is shaded.

The *relative complement* of a set B with respect to a set A or, simply, the *difference* between A and B , denoted by $A \setminus B$, is the set of elements which belong to A but which do not belong to B , that is,

$$A \setminus B = \{x : x \in A, x \notin B\}$$

The set $A \setminus B$ is read "A minus B". Some texts denote $A \setminus B$ by $A - B$ or $A \sim B$. Figure 1-4(b) is a Venn diagram in which $A \setminus B$ is shaded.

The *symmetric difference* of the sets A and B , denoted by $A \oplus B$, consists of those elements which belong to A or B , but not both. That is,

$$A \oplus B = (A \cup B) \setminus (A \cap B) \quad \text{or} \quad A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Figure 1-4(c) is a Venn diagram in which $A \oplus B$ is shaded.

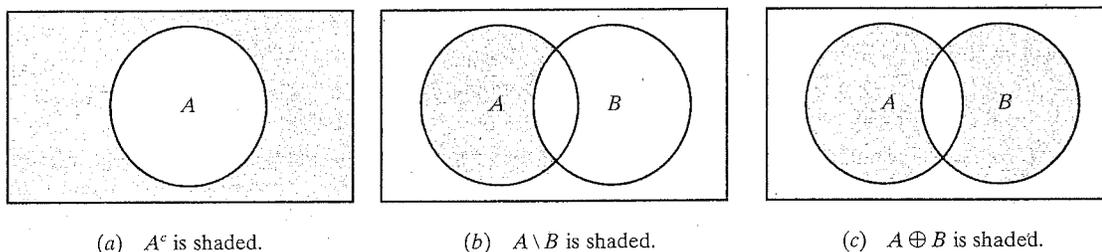


Fig. 1-4

EXAMPLE 1.3 Let $U = N = \{1, 2, 3, \dots\}$ be the universal set, and let

$$A = \{1, 2, 3, 4\}, \quad B = \{3, 4, 5, 6, 7\}, \quad C = \{2, 3, 8, 9\}, \quad E = \{2, 4, 6, \dots\}$$

[Here E is the set of even positive integers.] Then

$$A^c = \{5, 6, 7, \dots\}, \quad B^c = \{1, 2, 8, 9, 10, \dots\}, \quad E^c = \{1, 3, 5, \dots\}$$

That is, E^c is the set of odd integers. Also

$$\begin{array}{llll} A \setminus B = \{1, 2\}, & A \setminus C = \{1, 4\}, & B \setminus C = \{4, 5, 6, 7\}, & A \setminus E = \{1, 3\}, \\ B \setminus A = \{5, 6, 7\}, & C \setminus A = \{8, 9\}, & C \setminus B = \{2, 8, 9\}, & E \setminus A = \{6, 8, 10, \dots\} \end{array}$$

Furthermore

$$A \oplus B = (A \setminus B) \cup (B \setminus A) = \{1, 2, 5, 6, 7\}, \quad B \oplus C = \{2, 4, 5, 6, 7, 8, 9\},$$

$$A \oplus C = (A \setminus C) \cup (C \setminus A) = \{1, 4, 8, 9\}, \quad A \oplus E = \{1, 3, 6, 8, 10, \dots\}$$

Algebra of Sets

Sets under the operations of union, intersection, and complement satisfy various laws (identities) which are listed in Table 1-1. In fact, we formally state:

Theorem 1.5: Sets satisfy the laws in Table 1-1.

Table 1-1 Laws of the Algebra of Sets

Idempotent Laws	
1a. $A \cup A = A$	1b. $A \cap A = A$
Associative Laws	
2a. $(A \cup B) \cup C = A \cup (B \cup C)$	2b. $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative Laws	
3a. $A \cup B = B \cup A$	3b. $A \cap B = B \cap A$
Distributive Laws	
4a. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	4b. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	
5a. $A \cup \emptyset = A$	5b. $A \cap U = A$
6a. $A \cup U = U$	6b. $A \cap \emptyset = \emptyset$
Involution Law	
7. $(A^c)^c = A$	
Complement Laws	
8a. $A \cup A^c = U$	8b. $A \cap A^c = \emptyset$
9a. $U^c = \emptyset$	9b. $\emptyset^c = U$
DeMorgan's Laws	
10a. $(A \cup B)^c = A^c \cap B^c$	10b. $(A \cap B)^c = A^c \cup B^c$

Remark: Each law in Table 1-1 follows from an equivalent logical law. Consider, for example, the proof of DeMorgan's law:

$$(A \cup B)^c = \{x : x \notin (A \text{ or } B)\} = \{x : x \notin A \text{ and } x \notin B\} = A^c \cap B^c$$

Here we use the equivalent (DeMorgan's) logical law:

$$\neg(p \vee q) = \neg p \wedge \neg q$$

where \neg means "not", \vee means "or", and \wedge means "and". (Sometimes Venn diagrams are used to illustrate the laws in Table 1-1 as in Problem 1.17.)

Duality

The identities in Table 1-1 are arranged in pairs, as, for example, 2a and 2b. We now consider the principle behind this arrangement. Let E be an equation of set algebra. The *dual* E^* of E is the equation obtained by replacing each occurrence of $\cup, \cap, \mathbf{U}, \emptyset$ in E by $\cap, \cup, \emptyset, \mathbf{U}$, respectively. For example, the dual of

$$(\mathbf{U} \cap A) \cup (B \cap A) = A \quad \text{is} \quad (\emptyset \cup A) \cap (B \cup A) = A$$

Observe that the pairs of laws in Table 1-1 are duals of each other. It is a fact of set algebra, called the *principle of duality*, that, if any equation E is an identity, then its dual E^* is also an identity.

1.5 FINITE AND COUNTABLE SETS

Sets can be finite or infinite. A set S is *finite* if S is empty or if S consists of exactly m elements where m is a positive integer; otherwise S is infinite.

EXAMPLE 1.4

- (a) Let A denote the letters in the English alphabet, and let D denote the days of the week, that is, let

$$A = \{a, b, c, \dots, y, z\} \quad \text{and} \quad D = \{\text{Monday, Tuesday, } \dots, \text{Sunday}\}$$

Then A and D are finite sets. Specifically, A has 26 elements and D has 7 elements.

- (b) Let $R = \{x : x \text{ is a river on the earth}\}$. Although it may be difficult to count the number of rivers on the earth, R is still a finite set.

- (c) Let E be the set of even positive integers, and let \mathbf{I} be the *unit interval*; that is, let

$$E = \{2, 4, 6, \dots\} \quad \text{and} \quad \mathbf{I} = [0, 1] = \{x : 0 \leq x \leq 1\}$$

Then both E and \mathbf{I} are infinite sets.

Countable Sets

A set S is *countable* if S is finite or if the elements of S can be arranged in the form of a sequence, in which case S is said to be *countably infinite*. A set is *uncountable* if it is not countable. The above set E of even integers is countably infinite, whereas it can be proven that the unit interval $\mathbf{I} = [0, 1]$ is uncountable.

1.6 COUNTING ELEMENTS IN FINITE SETS, INCLUSION-EXCLUSION PRINCIPLE

The notation $n(S)$ or $|S|$ will denote the number of elements in a set S . Thus $n(A) = 26$ where A consists of the letters in the English alphabet, and $n(D) = 7$ where D consists of the days of the week. Also $n(\emptyset) = 0$, since the empty set has no elements.

The following lemma applies.

Lemma 1.6: Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and

$$n(A \cup B) = n(A) + n(B)$$

This lemma may be restated as follows:

Lemma 1.6: Suppose S is the disjoint union of finite sets A and B . Then S is finite and

$$n(S) = n(A) + n(B)$$

Proof: In counting the elements of $A \cup B$, first count the elements of A . There are $n(A)$ of these. The only other elements in $A \cup B$ are those that are in B but not in A . Since A and B are disjoint, no element of B is in A . Thus, there are $n(B)$ elements which are in B but not in A . Accordingly, $n(A \cup B) = n(A) + n(B)$.

For any sets A and B , the set A is the disjoint union of $A \setminus B$ and $A \cap B$ (Problem 1.45). Thus, Lemma 1.6 gives us the following useful result.

Corollary 1.7: Let A and B be finite sets. Then

$$n(A \setminus B) = n(A) - n(A \cap B)$$

That is, the number of elements in A but not in B is the number of elements in A minus the number of elements in both A and B . For example, suppose an art class A has 20 students and 8 of the students are also taking a biology class B . Then there are

$$20 - 8 = 12$$

students in the class A which are not in the class B .

Given any set A , we note that the universal set U is the disjoint union of A and A^c . Accordingly, Lemma 1.6 also gives us the following result.

Corollary 1.8: Suppose A is a subset of a finite universal set U . Then

$$n(A^c) = n(U) - n(A)$$

For example, suppose a class U of 30 students has 18 full-time students. Then there are

$$30 - 18 = 12$$

part-time students in the class.

Inclusion-Exclusion Principle

There is also a formula for $n(A \cup B)$, even when they are not disjoint, called the *inclusion-exclusion principle*. Namely,

Theorem (Inclusion-Exclusion Principle) 1.9: Suppose A and B are finite sets. Then $A \cap B$ and $A \cup B$ are finite and

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

That is, we find the number of elements in A or B (or both) by first adding $n(A)$ and $n(B)$ (inclusion) and then subtracting $n(A \cap B)$ (exclusion) since its elements were counted twice.

We can apply this result to get a similar result for three sets.

Corollary 1.10: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

Mathematical induction (Section 1.9) may be used to further generalize this result to any finite number of finite sets.

EXAMPLE 1.5 Suppose list A contains the 30 students in a mathematics class and list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students:

- (a) Only on list A
- (b) Only on list B
- (c) On list A or B (or both)
- (d) On exactly one of the two lists

- (a) List A contains 30 names and 20 of them are on list B ; hence $30 - 20 = 10$ names are only on list A . That is, by Corollary 1.7,

$$n(A \setminus B) = n(A) - n(A \cap B) = 30 - 20 = 10$$

- (b) Similarly, there are $35 - 20 = 15$ names only on list B . That is,

$$n(B \setminus A) = n(B) - n(A \cap B) = 35 - 20 = 15$$

- (c) We seek $n(A \cup B)$. Note we are given that $n(A \cap B) = 20$.

One way is to use the fact that $A \cup B$ is the disjoint union of $A \setminus B$, $A \cap B$, and $B \setminus A$ (Problem 1.54), which is pictured in Fig. 1-5 where we have also inserted the number of elements in each of the three sets $A \setminus B$, $A \cap B$, $B \setminus A$. Thus

$$n(A \cup B) = 10 + 20 + 15 = 45$$

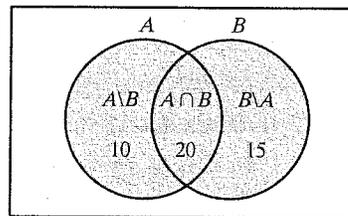
Alternately, by Theorem 1.8,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45$$

In other words, we combine the two lists and then cross out the 20 names which appear twice.

- (d) By (a) and (b), there are $10 + 15 = 25$ names on exactly one of the two lists; so $n(A \oplus B) = 25$. Alternately, by the Venn diagram in Fig. 1-5, there are 10 elements in $A \setminus B$, and 15 elements in $B \setminus A$; hence

$$n(A \oplus B) = 10 + 15 = 25$$



$A \cup B$ is shaded.

Fig. 1-5

1.7 PRODUCT SETS

Consider two arbitrary sets A and B . The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B . A short designation of this product is $A \times B$, which is read "A cross B". By definition,

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

One frequently writes A^2 instead of $A \times A$.

We note that ordered pairs (a, b) and (c, d) are equal if and only if their *first* elements, a and c , are equal and their *second* elements, b and d , are equal. That is,

$$(a, b) = (c, d) \quad \text{if and only if} \quad a = c \text{ and } b = d$$

EXAMPLE 1.6 \mathbf{R} denotes the set of real numbers, and so $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$ is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of \mathbf{R}^2 as points in the plane, as in Fig. 1-6. Here each point P represents an ordered pair (a, b) of real numbers, and vice versa; the vertical line through P meets the x axis at a , and the horizontal line through P meets the y axis at b . \mathbf{R}^2 is frequently called the *Cartesian plane*.

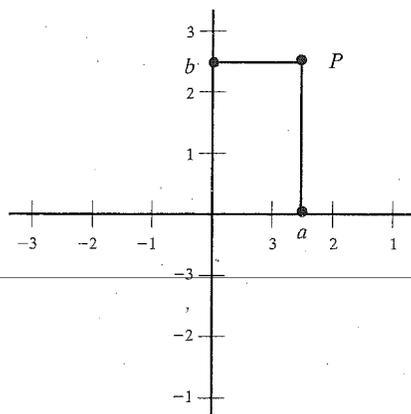
Cartesian Plane R^2

Fig. 1-6

EXAMPLE 1.7 Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then

$$A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$$

$$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

Also,

$$A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

There are two things worth noting in the above Example 1.7. First of all, $A \times B \neq B \times A$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important.

Secondly, using $n(S)$ for the number of elements in a set S , we have:

$$n(A \times B) = 6 = 2 \cdot 3 = n(A) \cdot n(B)$$

In fact, $n(A \times B) = n(A) \cdot n(B)$ for any finite sets A and B . This follows from the observation that, for each $a \in A$, there will be $n(B)$ ordered pairs in $A \times B$ beginning with a . Hence, altogether there will be $n(A)$ times $n(B)$ ordered pairs in $A \times B$.

We state the above result formally.

Theorem 1.11: Suppose A and B are finite. Then $A \times B$ is finite and

$$n(A \times B) = n(A) \cdot n(B)$$

The concept of a product of sets can be extended to any finite number of sets in a natural way. That is, for any sets A_1, A_2, \dots, A_m , the set of all ordered m -tuples (a_1, a_2, \dots, a_m) , where $a_1 \in A_1$, $a_2 \in A_2, \dots, a_m \in A_m$, is called the *product* of the sets A_1, A_2, \dots, A_m and is denoted by

$$A_1 \times A_2 \times \cdots \times A_m \quad \text{or} \quad \prod_{i=1}^m A_i$$

Just as we write A^2 instead of $A \times A$, so we write A^m for $A \times A \times \cdots \times A$, where there are m factors.

Furthermore, for finite sets A_1, A_2, \dots, A_m , we have

$$n(A_1 \times A_2 \times \cdots \times A_m) = n(A_1)n(A_2) \cdots n(A_m)$$

That is, Theorem 1.11 may be easily extended, by induction, to the product of m sets.

1.8 CLASSES OF SETS, POWER SETS, PARTITIONS

Given a set S , we may wish to talk about some of its subsets. Thus, we would be considering a "set of sets". Whenever such a situation arises, to avoid confusion, we will speak of a *class* of sets or a *collection* of sets. The words "subclass" and "subcollection" have meanings analogous to subset.

EXAMPLE 1.8 Suppose $S = \{1, 2, 3, 4\}$. Let \mathcal{A} be the class of subsets of S which contains exactly three elements of S . Then

$$\mathcal{A} = [\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}]$$

The elements of \mathcal{A} are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$.

Let \mathcal{B} be the class of subsets of S which contains the numeral 2 and two other elements of S . Then

$$\mathcal{B} = [\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}]$$

The elements of \mathcal{B} are $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{2, 3, 4\}$. Thus \mathcal{B} is a subclass of \mathcal{A} . (To avoid confusion, we will usually enclose the sets of a class in brackets instead of braces.)

Power Sets

For a given set S , we may consider the class of all subsets of S . This class is called the *power set* of S , and it will be denoted by $\mathcal{P}(S)$. If S is finite, then so is $\mathcal{P}(S)$. In fact, the number of elements in $\mathcal{P}(S)$ is 2 raised to the power of S ; that is,

$$n(\mathcal{P}(S)) = 2^{n(S)}$$

(For this reason, the power set of S is sometimes denoted by 2^S .) We emphasize that S and the empty set \emptyset belong to $\mathcal{P}(S)$ since they are subsets of S .

EXAMPLE 1.9 Suppose $S = \{1, 2, 3\}$. Then

$$\mathcal{P}(S) = [\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S]$$

As expected from the above remark, $\mathcal{P}(S)$ has $2^3 = 8$ elements.

Partitions

Let S be a nonempty set. A *partition* of S is a *subdivision* of S into nonoverlapping, nonempty subsets. Precisely, a *partition* of S is a collection $\{A_i\}$ of nonempty subsets of S such that

- (i) Each a in S belongs to one of the A_i .
- (ii) The sets of $\{A_i\}$ are mutually disjoint; that is, if

$$A_i \neq A_j, \text{ the } A_i \cap A_j = \emptyset.$$

The subsets in a partition are called *cells*. Figure 1-7 is a Venn diagram of a partition of the rectangular set S of points into five cells, A_1, A_2, A_3, A_4, A_5 .

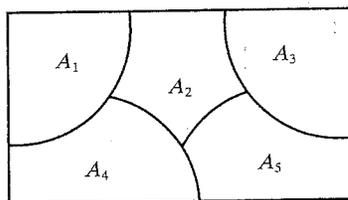


Fig. 1-7

EXAMPLE 1.10 Consider the following collections of subsets of $S = \{1, 2, 3, \dots, 8, 9\}$:

- (i) $\{\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}\}$
- (ii) $\{\{1, 3, 5\}, \{2, 4, 6, 8\}, \{5, 7, 9\}\}$
- (iii) $\{\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}\}$

Then (i) is not a partition of S since 7 in S does not belong to any of the subsets. Furthermore, (ii) is not a partition of S since $\{1, 3, 5\}$ and $\{5, 7, 9\}$ are not disjoint. On the other hand, (iii) is a partition of S .

Indexed Classes of Sets

An *indexed class of sets*, usually presented in the form

$$\{A_i : i \in I\} \quad \text{or simply} \quad \{A_i\}$$

means that there is a set A_i assigned to each element $i \in I$. The set I is called the *indexing set* and the sets A_i are said to be indexed by I . The *union* of the sets A_i , written $\cup_{i \in I} A_i$ or simply $\cup_i A_i$, consists of those elements which belong to at least one of the A_i ; and the *intersection* of the sets A_i , written $\cap_{i \in I} A_i$ or simply $\cap_i A_i$, consists of those elements which belong to every A_i .

When the indexing set is the set \mathbf{N} of positive integers, the indexed class $\{A_1, A_2, \dots\}$ is called a *sequence* of sets. In such a case, we also write

$$\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \dots \quad \text{and} \quad \cap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \dots$$

for the union and intersection, respectively, of a sequence of sets.

Definition: A nonempty class \mathcal{A} of subsets of \mathbf{U} is called an *algebra* (σ -*algebra*) of sets if it has the following two properties:

- (i) The complement of any set in \mathcal{A} belongs to \mathcal{A} .
- (ii) The union of any finite (countable) number of sets in \mathcal{A} belongs to \mathcal{A} .

That is, \mathcal{A} is closed under complements and finite (countable) unions.

It is simple to show (Problem 1.40) that any algebra (σ -algebra) of sets contains \mathbf{U} and \emptyset and is closed under finite (countable) intersections.

1.9 MATHEMATICAL INDUCTION

An essential property of the set $\mathbf{N} = \{1, 2, 3, \dots\}$ of positive integers which is used in many proofs follows:

Principle of Mathematical Induction I: Let $A(n)$ be an assertion about the set \mathbf{N} of positive integers, that is, $A(n)$ is true or false for each integer $n \geq 1$. Suppose $A(n)$ has the following two properties:

- (i) $A(1)$ is true.
- (ii) $A(n + 1)$ is true whenever $A(n)$ is true.

Then $A(n)$ is true for every positive integer.

We shall not prove this principle. In fact, this principle is usually given as one of the axioms when \mathbf{N} is developed axiomatically.

EXAMPLE 1.11 Let $A(n)$ be the assertion that the sum of the first n odd numbers is n^2 ; that is,

$$A(n): 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

(The n th odd number is $2n - 1$ and the next odd number is $2n + 1$.)

Observe that $A(n)$ is true for $n = 1$ since

$$A(1): 1 = 1^2$$

Assuming $A(n)$ is true, we add $2n + 1$ to both sides of $A(n)$, obtaining

$$1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2$$

However, this is $A(n + 1)$. That is, $A(n + 1)$ is true assuming $A(n)$ is true. By the principle of mathematical induction, $A(n)$ is true for all $n \geq 1$.

There is another form of the principle of mathematical induction which is sometimes more convenient to use. Although it appears different, it is really equivalent to the above principle of induction.

Principle of Mathematical Induction II: Let $A(n)$ be an assertion about the set \mathbf{N} of positive integers with the following two properties:

- (i) $A(1)$ is true.
- (ii) $A(n)$ is true whenever $A(k)$ is true for $1 \leq k \leq n$.

Then $A(n)$ is true for every positive integer.

Remark: Sometimes one wants to prove that an assertion A is true for a set of integers of the form

$$\{a, a + 1, a + 2, \dots\}$$

where a is any integer, possibly 0. This can be done by simply replacing 1 by a in either of the above Principles of Mathematical Induction.

Solved Problems

SETS, ELEMENTS, SUBSETS

1.1. List the elements of the following sets; here $\mathbf{N} = \{1, 2, 3, \dots\}$:

- (a) $A = \{x : x \in \mathbf{N}, 2 < x < 9\}$,
 - (b) $B = \{x : x \in \mathbf{N}, x \text{ is even}, x \leq 15\}$,
 - (c) $C = \{x : x \in \mathbf{N}, x + 5 = 2\}$,
 - (d) $D = \{x : x \in \mathbf{N}, x \text{ is a multiple of } 5\}$
- (a) A consists of the positive integers between 2 and 9; hence $A = \{3, 4, 5, 6, 7, 8, 9\}$.
- (b) B consists of the even positive integers less than or equal to 15; hence $B = \{2, 4, 6, 8, 10, 12, 14\}$.
- (c) There are no positive integers which satisfy the condition $x + 5 = 2$; hence C contains no elements. In other words, $C = \emptyset$, the empty set.
- (d) D is infinite, so we cannot list all its elements. However, sometimes we write $D = \{5, 10, 15, 20, \dots\}$ assuming everyone understands that we mean the multiples of 5.

1.2. Which of these sets are equal: $\{r, s, t\}$, $\{t, s, r\}$, $\{s, r, t\}$, $\{t, r, s\}$?

They are all equal. Order does not change a set.