

Probability in Computing



Reminders

- HW 5 due on Thursday **Reading**
- LLM 18.7, 18.8

LECTURE 10

Last time

- Conditional Probability
- Tree Diagrams
- Product Rule
- Independent Events

Today

- Independent Events
- Sequences of Independent Events
- Bayes' Rule

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MIDTERM ANNOUNCEMENT

- Midterm will be Thursday, March 3rd, 6:30 8:30 in CGS 129
- Closed book, no devices.
- One hand-written 1-sided page of notes allowed on colored paper, significantly different from white.
- All material up to tomorrow's discussion (Friday, 2/25)
- Practice Exam will be distributed soon; Friday discussion will be practice problems, next Tuesday lecture will be review.
- Review TopHat, Homework, and Discussion Problems, plus exercises suggested from supplimentary textbook.



Definition: Independent Events

Two events A and B are independent if $Pr(A \cap B) = Pr(A) \cdot Pr(B)$

- Equivalent condition: Pr(A | B) = Pr(A)Alternatively, Pr(B | A) = Pr(B)
- Independence is symmetric: if *A* is independent of *B* then *B* is independent of *A*
- This definition can be used even when Pr(B) = 0
 - An event B with Pr(B) = 0 is independent of all events, including itself.



Conditioning the original sample space means changing the perspective: when A and B are independent, then Pr(A) does not change when reduce the sample space from Ω to B:





Be Careful: Independent is not the same as disjoint!

When events A and B are disjoint, Pr(A), Pr(B) > 0, if A happens then B is guaranteed not to happen and vice versa:



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Let's try some examples.....

Experiment: roll two fair 4-sided dice A = "first die is 1" B = "sum of the two rolls is 5" Are A and B independent?

Experiment: roll two fair 4-sided dice A = "first die is 1" B = "sum of the two rolls is ≤ 5 " Are *A* and *B* independent?



Experiment: toss two fair coins

- A = "first toss is H"
- B = "both tosses give the same result" Are *A* and *B* independent?
- A. YES
- B. NO

CS Top Hat question (Join Code: 413437)

Experiment: toss two biased coins with Pr(H) = 2/3

- A = "first toss is H"
- B = "both tosses give the same result"

Are A and B independent?

- A. YES
- B. NO

Independent Repeated Trials: Example

- Experiment: X = spin the dial of the spinner and observe the region where it stopped.
 - $\Omega = \{1, 2, 3, 4\}$
 - $\Pr(1) = \frac{1}{2}$, $\Pr(2) = \frac{1}{4}$, $\Pr(3) = \Pr(4) = \frac{1}{8}$
- Now we spin the dial twice (X_1, X_2)
 - Sample space is $\Omega \times \Omega = \{(i, j) : 1 \le i, j \le 4\}$
 - $Pr((i,j)) = Pr(i) \cdot Pr(j)$
- We can spin it *n* times
 - Each outcome is a sequence of n elements, each drawn from Ω

 $(X_1, X_2, \ldots X_n)$

- Sample space is Ω^n
- $\Pr((s_1, \dots, s_n)) = \Pr(s_1) \cdot \dots \cdot \Pr(s_n)$

Technical note: We are overusing PrWe are talking about twodifferent probability functions(over Ω^n and over Ω)

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also denoted Ω^2

Events ``1st spin is *i*'' and ``2nd spin is *j*'' are independent



Independent Repeated Trials

- Let Ω be a sample space for X with a probability function Pr and $n \in \mathbb{N}$.
- Let Ω^n denote the set of all length-*n* sequences of elements from Ω .
- Then n independent repeated trials of the original random experiment are represented by n random variables (X_1, X_2, \dots, X_n) with
 - sample space Ω^n
 - probability function $Pr((s_1, ..., s_n)) = Pr(s_1) \cdot ... \cdot Pr(s_n)$

Examples

- Roll a 6-sided die twice: (X_1, X_2)
 - Sample space for one roll is $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Pr(s) = \frac{1}{6}$ for all $s \in \Omega$
 - Sample space for two rolls is Ω^2 , and $Pr((s_1, s_2)) = \frac{1}{36}$ for all $s_1, s_2 \in \Omega$
- Toss a fair coin until you see a head: $(X_1, X_2, \dots, X_k, \dots)$
 - Sample space for one toss is $\Omega = \{0,1\}$ and $Pr(0) = Pr(1) = \frac{1}{2}$
 - Sample space for one trial is $\Omega^{\mathbb{N}}$; an outcome in Ω^k has probability $\frac{1}{2^k}$.



P(A | B) considers an event B followed by an event A, and how the occurence of B affects the occurence of A. What are the labels on a tree diagram of this random experiment?





How does this relate to tree diagrams?

When the events are independent, then we have the familiar tree diagram in which we simply write the probabilities of the events on each arc:





Example: Sampling with or without replacement

1. You draw 3 cards from a standard deck with replacement: what is the probability they are all Spades?

2. You draw 3 cards from a standard deck without replacement: what is the probability they are all Spades?

Which of these represents three different random variables? (X, Y, Z)

Which of these represents independent trials of the same X? (X_1, X_2, X_3)





Example: Sampling with or without replacement

1. You draw 3 cards from a standard deck with replacement: what is the probability they are all Spades?

$$\frac{13}{52} \cdot \frac{13}{52} \cdot \frac{13}{52} \qquad (X_1, X_2, X_3)$$

2. You draw 3 cards from a standard deck without replacement: what is the probability they are all Spades?

$$\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50}$$
 (X, Y, Z)



Fun fact: Dependence does not imply causality!

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We can rearrange the conditional probability rule in a way that makes the sequence of the events irrelevant -- which happened first, A or B? Or did they happen at the same time? Does it matter? $P(A \cap B) = P(B \cap A)$

$$P(A | B) = \frac{P(A \cap B)}{P(B)} \qquad \qquad P(B | A) = \frac{P(B \cap A)}{P(A)}$$

We can do a little algebra to define conditional probabilities in terms of each other:

$$P(B \mid A) * P(A) = P(B \cap A) = P(A \mid B) * P(B)$$

so:

$$P(B \mid A) = \frac{P(A \mid B) * P(B)}{P(A)}$$



This has an interesting flavor, because we can ask about causes of outcomes:

A Priori Reasoning -- "I randomly choose a person and observe that he is male; what the probability that it is a smoker?"

"The first toss of a pair of dice is a 5; what is the probability that the total is greater than 8?"

A Posteriori Reasoning -- "I find a cigarette butt on the ground, what is the probability that it was left by a man?"

"The total of a pair of thrown dice is greater than 8; what is the probability that the first toss was a 5?"



The best way to understand this is to view it with a tree diagram! P(B|A) = the probability that when A happens, it was "preceeded" by B: If A has happened, what is the



If A has happened, what is the probability that it did so on the path where B also occurred?

Note:

$$A = P(A \cap B) \cup P(A \cap B^{c})$$

So what percentage of A is due to $A \cap B$?

Same calculation as:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}$$



Example: Suppose that 10% of female BU students smoke cigarettes and 20% of male BU students smoke cigarettes.Suppose that 60% of BU students are female and 40% male.I see a cigarette butt on the pavement. What is the probability it was thrown there by a female student?



Example: We have two urns, A and B. A contains 2 red balls and 3 black balls. Urn B contains 2 red balls and 1 black ball. We select an urn uniformly at random and pick a ball and find it is red. What is the probability it came from Urn A?