2D Motion Analysis using Optical Flow

Margrit Betke, CS 585, Spring 2024

Some slides adapted from E. Learned-Miller, S. Lazebnik, S. Seitz, R. Szeliski, and M. Pollefeys

Motion

Goal: Understand motion in 3D world of

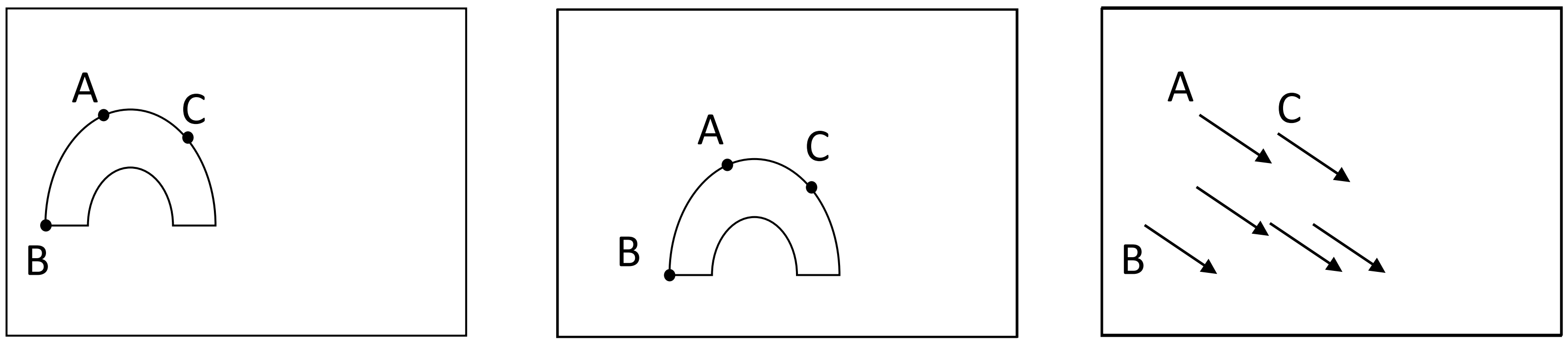
- rigid objects: translations and rotations
- non-rigid objects: deformations

Motion Field

assigns velocity vector to each object pixel in image

1. Translation

$E(x,y,t_0)$ $E(x,y,t_1)$ Motion Field



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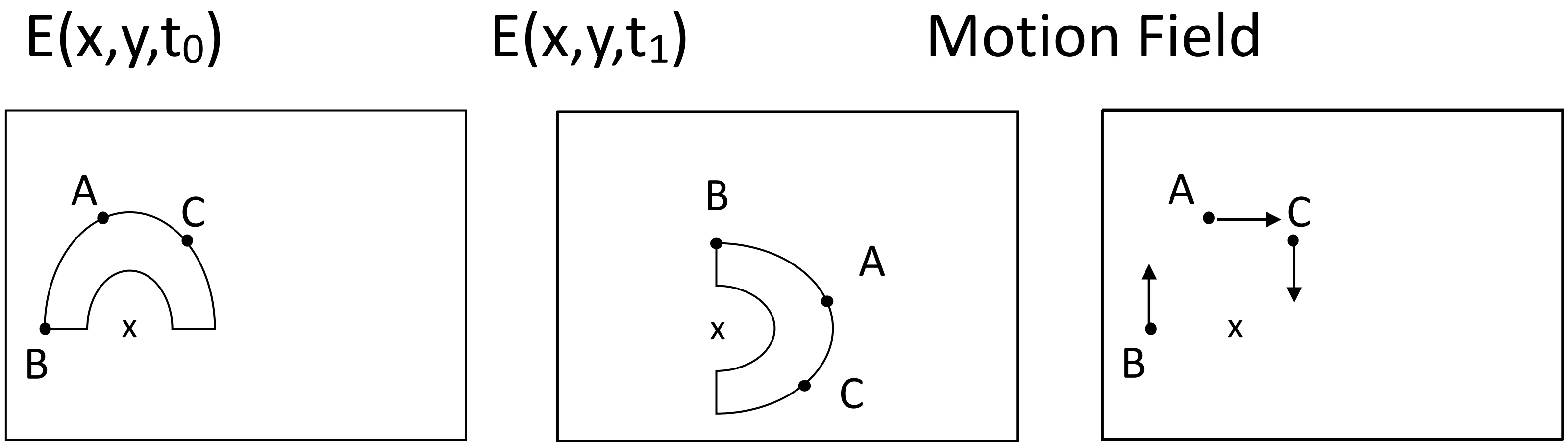
Motion & Optical Flow Fields

2. Rotation



Optical Flow

= apparent motion of brightness pattern

How does $E(x,y,t)$ change?

Hope: Brightness changes due to object motion.

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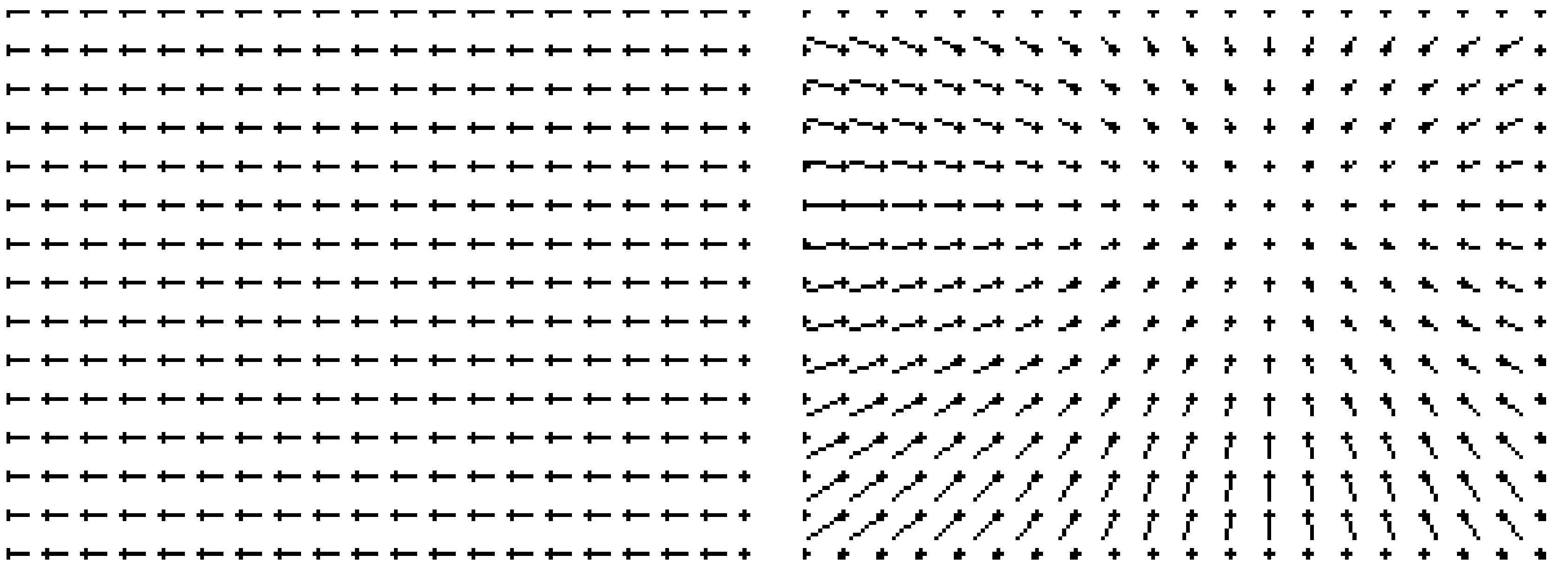
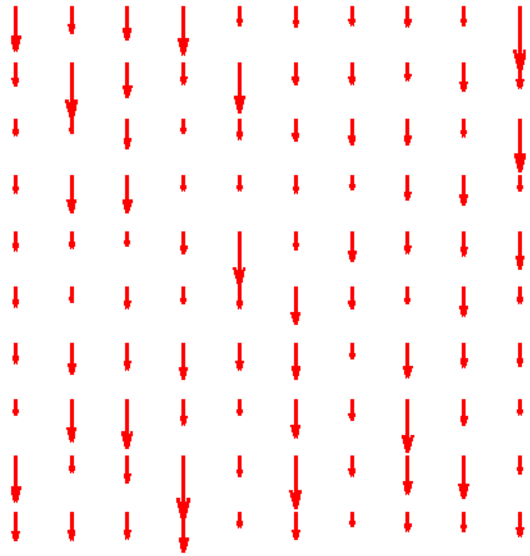
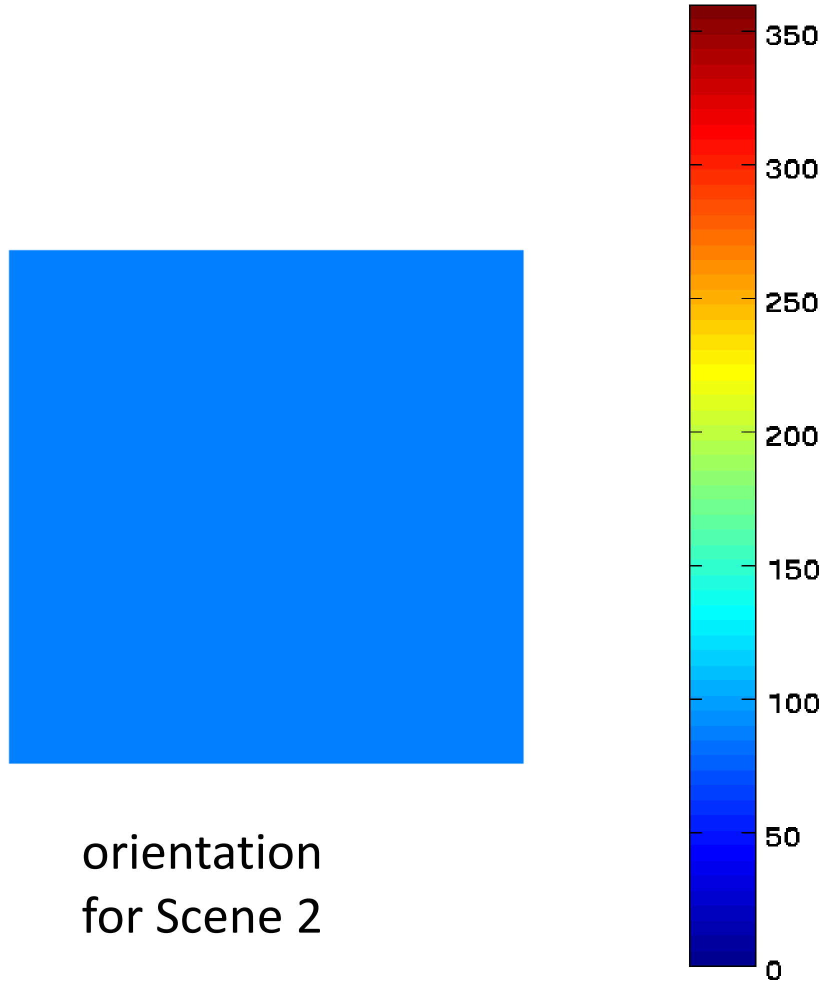
Optical Flow Field Examples



Motion along y axis



optical flow
for Scene 2



orientation
for Scene 2

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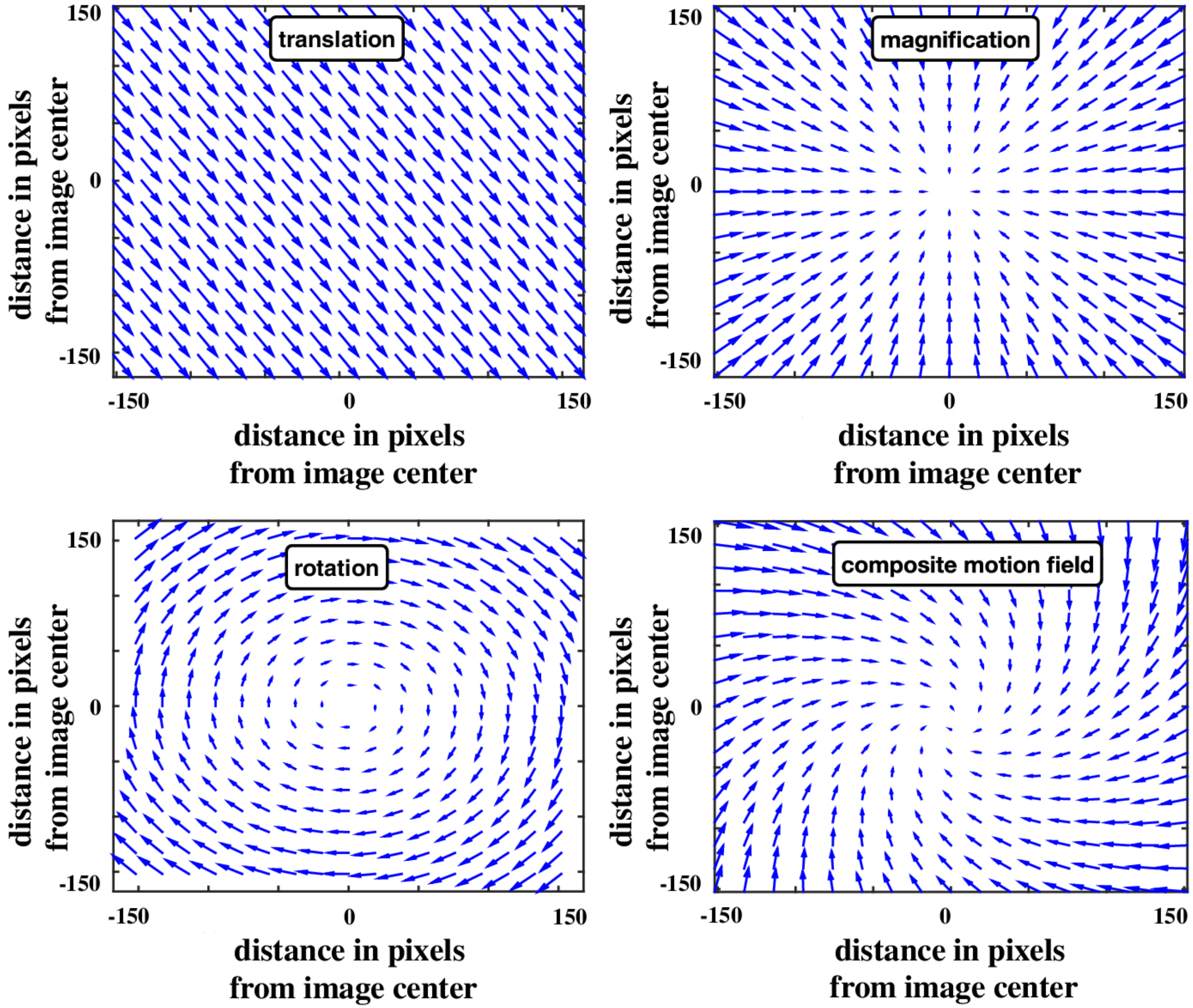

Citation

Brian J. Thelen, John R. Valenzuela, Joel W. LeBlanc, "Theoretical performance assessment and empirical analysis of super-resolution under unknown affine sensor motion," J. Opt. Soc. Am. A **33,** 519-526 (2016);
https://www.osapublishing.org/josaa/abstract.cfm?uri=josaa-33-4-519

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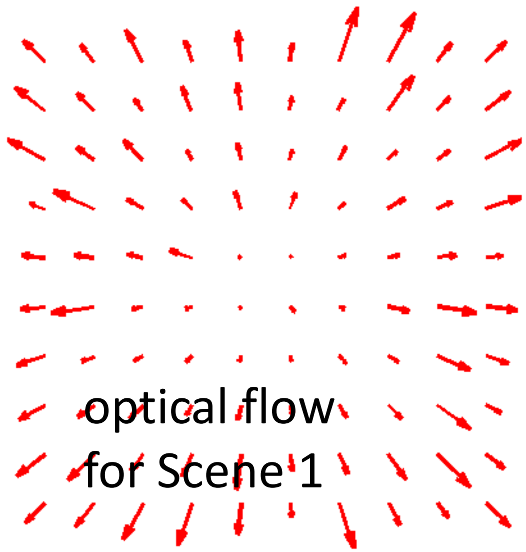
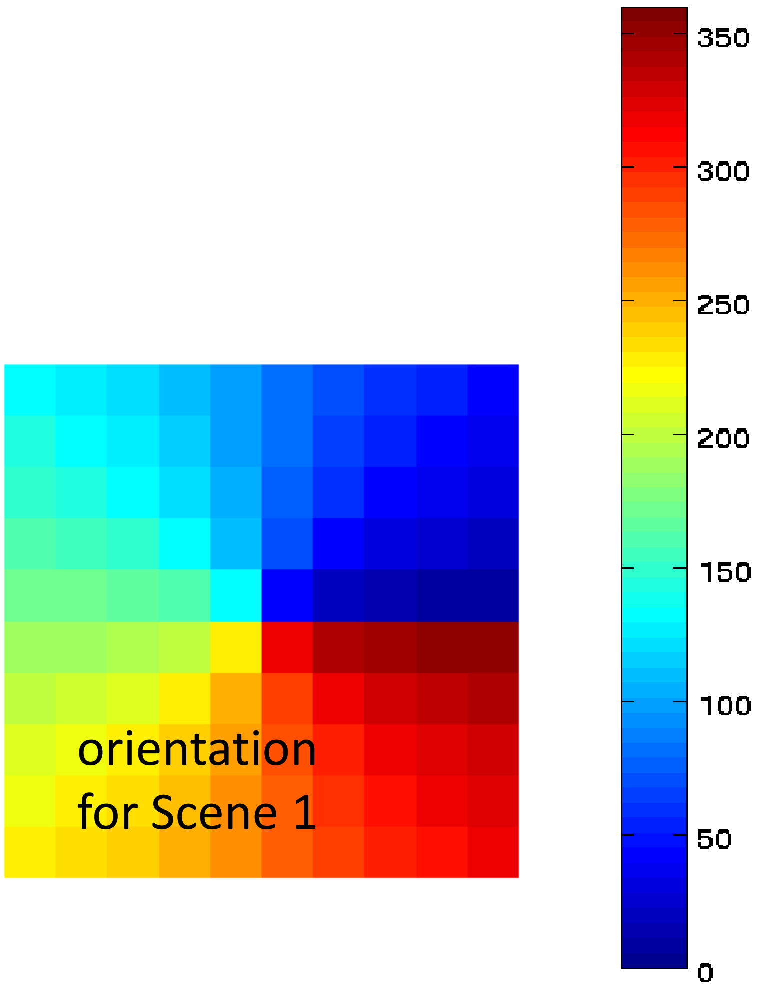
Motion along z axis





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Optical Flow and Motion Fields

- Definition: Optical flow is the *apparent* motion of brightness patterns in the image
- Ideal case: optical flow = motion field
- Warning: Apparent motion can be caused by lighting changes without any actual motion
  - rotating sphere under fixed lighting (zero optical flow but non-zero motion field)
  - stationary sphere under moving illumination (non-zero optical flow but zero motion)

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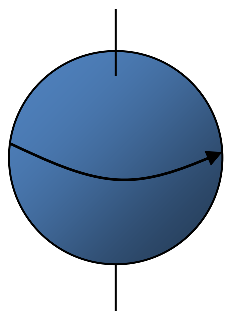
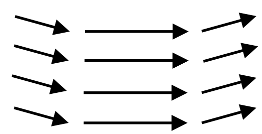
Motion & Optical Flow Fields

Examples:

1. Sphere rotating under constant illumination

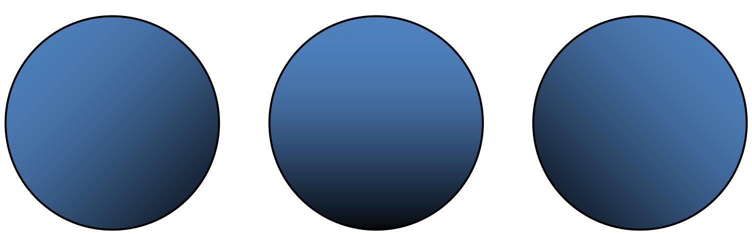


Motion Field



Optical Flow Field
zero

2. Fixed Sphere, light source moving



Motion Field
zero

Optical Flow Fields
not zero

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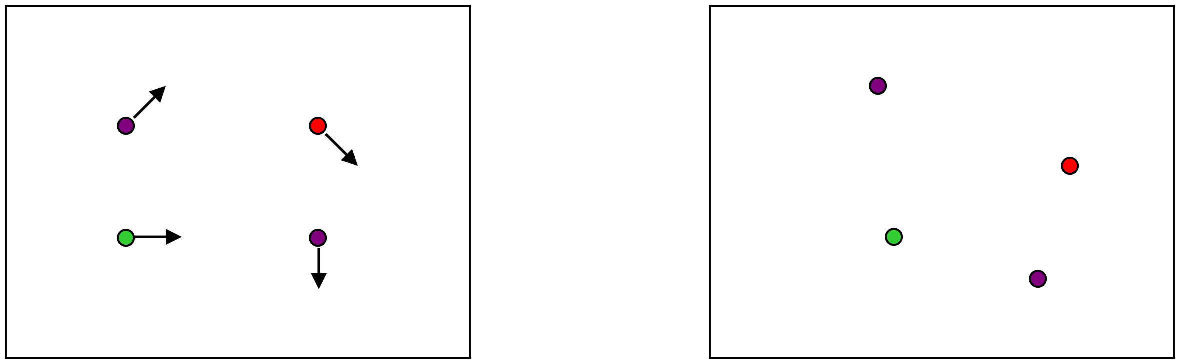
Estimating optical flow



$E(x, y, t)$ $\qquad$ $E(x+\delta x, y+\delta y, t+\delta t)$

Task: Given two subsequent frames, estimate the apparent motion field $u(x,y)$ and $v(x,y)$ between them

Key assumptions:

- **Brightness constancy:** projection of the same point looks the same in every frame (same gray value)
- **Small motion:** points do not move very far
- **Spatial coherence:** points move like their neighbors

Constant Brightness Assumption (CBA) in 1D

$$E(x + \delta x, t + \delta t) = E(x,t)$$

horizontal motion only

$\downarrow$ Taylor Series Expansion

$$E(x,t) + \delta x\, E_x + \delta t\, E_t = E(x,t)$$

E_x = partial derivative of E with respect to x

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Constant Brightness Assumption (CBA) in 1D

$$E(x + \delta x, t + \delta t) \quad = \quad E(x,t)$$

$\downarrow$ Taylor Series Expansion

$$E(x,t) + \delta x \, E_x + \delta t \, E_t \quad = E(x,t)$$

E_x = partial derivative of E with respect to x

$$\underbrace{\delta x / \delta t}_{\text{Horizontal velocity u at pixel x}} E_x + E_t = 0$$

$$u \, E_x + E_t = 0 \quad \text{or} \quad u = -E_t / E_x$$

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Constant Brightness Assumption (CBA) in 1D

$$E(x + \delta x, t + \delta t) = E(x,t)$$

$\downarrow$ Taylor Series Expansion

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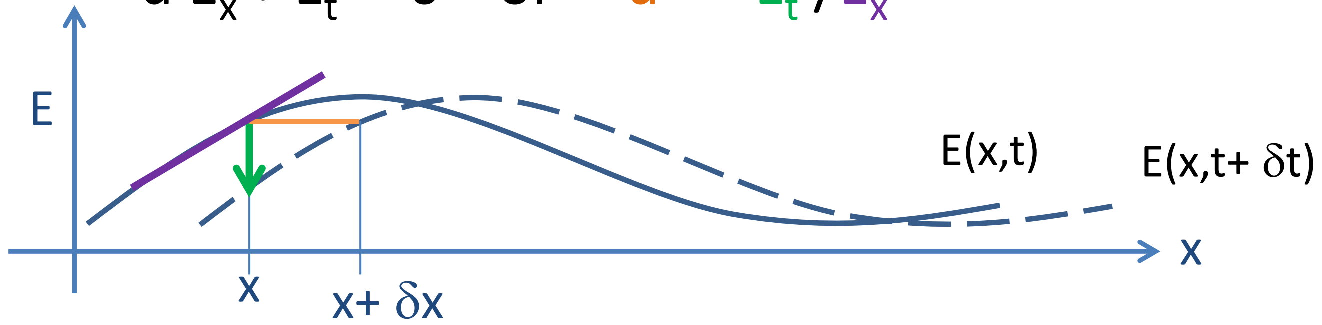

Constant Brightness Assumption (CBA) in 1D

Approximation for E_x : $$\frac{E(x+1,t) - E(x,t)}{\text{pixel width}}$$

E_t : $$\frac{E(x,t+\delta t) - E(x,t)}{\delta t}$$

$\delta t = 1/$ frame rate

$u = - E_t / E_x$ (for E_x not zero)

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Constant Brightness Assumption (CBA) in 1D

Approximation for E_x : $\dfrac{E(x+1,t) - E(x,t)}{\text{pixel width}}$

E_t : $\dfrac{E(x,t+\delta t) - E(x,t)}{\delta t}$

$\delta t = 1/$ frame rate

$u = - E_t / E_x$ (for E_x not zero)

Example:

	x-2	x-1	x	x+1	x+2	
	135	145	155	165	175	frame 1
	155	165	175	185	195	frame 2

Velocity u at pixel x = ?

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Constant Brightness Assumption (CBA) in 1D

Approximation for E_x : $$\frac{E(x+1,t) - E(x,t)}{\text{pixel width}}$$

E_t : $$\frac{E(x,t+\delta t) - E(x,t)}{\delta t}$$

$\delta t = 1/\text{ frame rate}$

$u = -E_t / E_x$ (for E_x not zero)

Example:

	x-2	x-1	x	x+1	x+2	
	135	145	155	165	175	frame 1
	155	165	175	185	195	frame 2

$$u = -\frac{175 - 155}{165 - 155} = -20/10 = -2$$

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Constant Brightness Assumption (CBA) in 1D

Approximation for E_x : $\dfrac{E(x+1,t) - E(x,t)}{\text{pixel width}}$

E_t : $\dfrac{E(x,t+\delta t) - E(x,t)}{\delta t}$

$\delta t = 1/$ frame rate

$u = - E_t / E_x$ (for E_x not zero)

Example:

	x-2	x-1	x	x+1	x+2	
	135	145	$\boxed{155}$	165	175	frame 1
	155	165	175	185	195	frame 2

$$u = -\frac{175 - 155}{165 - 155} = -20/10 = -2,$$

which means that the object at pixel x moves 2 pixels to the left per frame

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Constant Brightness Assumption (CBA) in 1D with Noisy Measurements

Rigid object motion, but brightness not everywhere constant

Determine patch P = image region with same constant velocity u

Use Least Squares Method to estimate u:

$$\min_u \sum_{i \in P} (uE_{x_i} + E_{t_i})^2$$

$$\frac{d}{du} \sum_{i \in P} (uE_{x_i} + E_{t_i})^2 = 0$$

$$u \sum_{i \in P} E_{x_i}^2 + \sum_{i \in P} E_{x_i} E_{t_i} = 0$$

$$u = -\frac{\sum_{i \in P} E_{x_i} E_{t_i}}{\sum_{i \in P} E_{x_i}^2}$$

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Constant Brightness Assumption (CBA) in 1D

Revisiting our example: $u = -E_t / E_x = -2$

x-1	x	x+1	x+2	
145	$\boxed{155}$	165	175	frame 1
165	175	185	195	frame 2

With noise:

x-1	x	x+1	x+2	
145	$\boxed{156}$	164	176	frame 1
165	173	184	193	frame 2

$$u = -\frac{\sum_{i \in P} E_{x_i} E_{t_i}}{\sum_{i \in P} E_{x_i}^2}$$

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Constant Brightness Assumption (CBA) in 1D

Revisiting our example: $u = -E_t / E_x = -2$

	x-1	x	x+1	x+2	
	145	$\boxed{155}$	165	175	frame 1
	165	175	185	195	frame 2

With noise:

	x-1	x	x+1	x+2	
	145	$\boxed{156}$	164	176	frame 1
	165	173	184	193	frame 2

$$u = -\frac{\sum_{i \in P} E_{x_i} E_{t_i}}{\sum_{i \in P} E_{x_i}^2}$$

$E_{x1} = 11$, $E_{x2} = 8$, $E_{x3} = 12$ and $E_{t1} = 20$, $E_{t2} = 17$, $E_{t3} = 20$

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Constant Brightness Assumption (CBA) in 1D

Revisiting our example: $u = - E_t / E_x = -2$

	x-1	x	x+1	x+2	
	145	$\boxed{155}$	165	175	frame 1
	165	175	185	195	frame 2

With noise:

$$u = -\frac{\sum_{i \in P} E_{x_i} E_{t_i}}{\sum_{i \in P} E_{x_i}^2}$$

	x-1	x	x+1	x+2	
	145	$\boxed{156}$	164	176	frame 1
	165	173	184	193	frame 2

$E_{x1} = 11$, $E_{x2} = 8$, $E_{x3} = 12$ and $E_{t1} = 20$, $E_{t2} = 17$, $E_{t3} = 20$

$$u = - \frac{220 + 136 + 240}{121 + 64 + 144} = - 594/329 \approx - 1.8$$

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Constant Brightness Assumption (CBA) in 1D

Revisiting our example: $u = -E_t / E_x = -2$

	x-1	x	x+1	x+2	
	145	$\boxed{155}$	165	175	frame 1
	165	175	185	195	frame 2

With noise:

	x-1	x	x+1	x+2	
	145	$\boxed{156}$	164	176	frame 1
	165	173	184	193	frame 2

$E_{x1} = 11$, $E_{x2} = 8$, $E_{x3} = 12$ and $E_{t1} = 20$, $E_{t2} = 17$, $E_{t3} = 20$

$$u = -\frac{220 + 136 + 240}{121 + 64 + 144} = -594/329 \approx -1.8$$

which means that the object at pixel x moves almost 2 pixels to the left per frame

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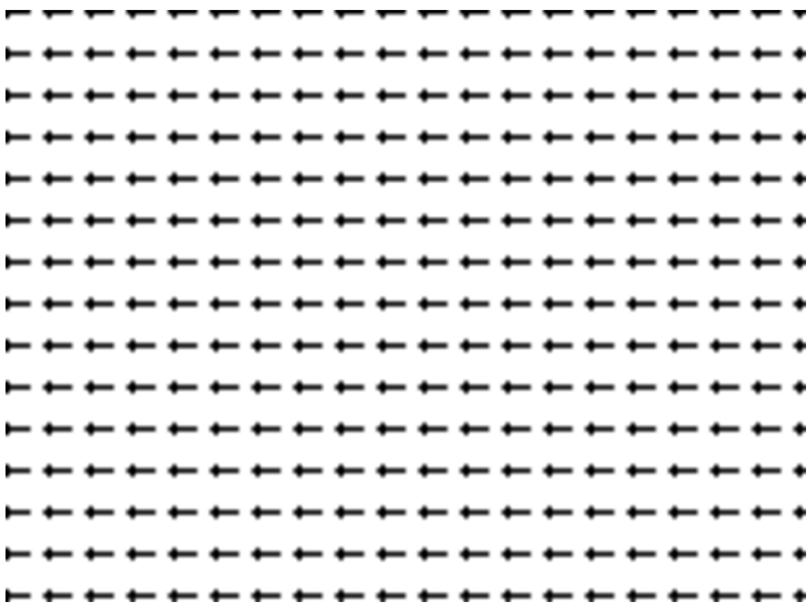
Optical Flow Field for u = 1.8



Velocity vector = $(u,v)^T = (-1.8, 0)^T$
Length of vectors = -1.8
Horizontal translation in negative direction

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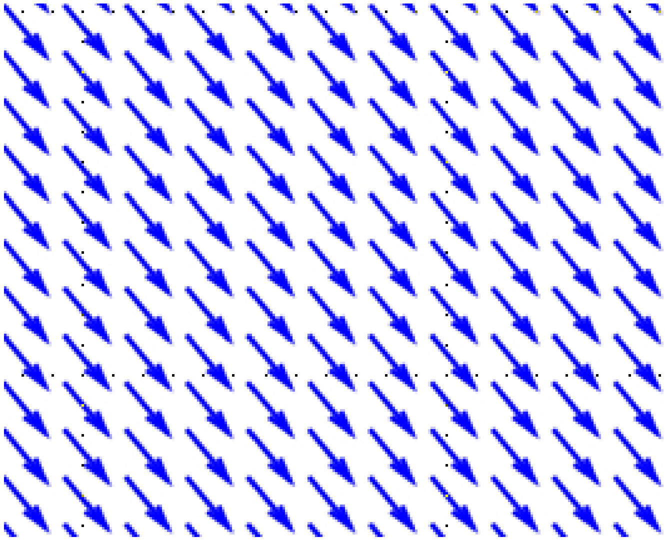
Optical Flow Field



Velocity vectors = $(u,v)^T = (0.5, -1)^T$

General Translation

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Constant Brightness Assumption (CBA) in 2D

$$E(x + \delta x, y + \delta y, t + \delta t) \quad = \; E(x,y,t)$$

$\downarrow$ Taylor Series Expansion

$$E(x,y,t) + \delta x\, E_x + \delta y\, E_y + \delta t\, E_t \qquad = E(x,y,t)$$

E_x = partial derivative of E with respect to x
E_y = partial derivative of E with respect to y

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Constant Brightness Assumption (CBA)

$$E(x + \delta x, y + \delta y, t + \delta t) = E(x,y,t)$$

$\downarrow$ Taylor Series Expansion

$$E(x,y,t) + \delta x\, E_x + \delta y\, E_y + \delta t\, E_t = E(x,y,t)$$

$$dx/dt\, E_x + dy/dt\, E_y + E_t = 0$$

$$\boxed{u\, E_x + v\, E_y + E_t = 0}$$

Constant Brightness Assumption (CBA)

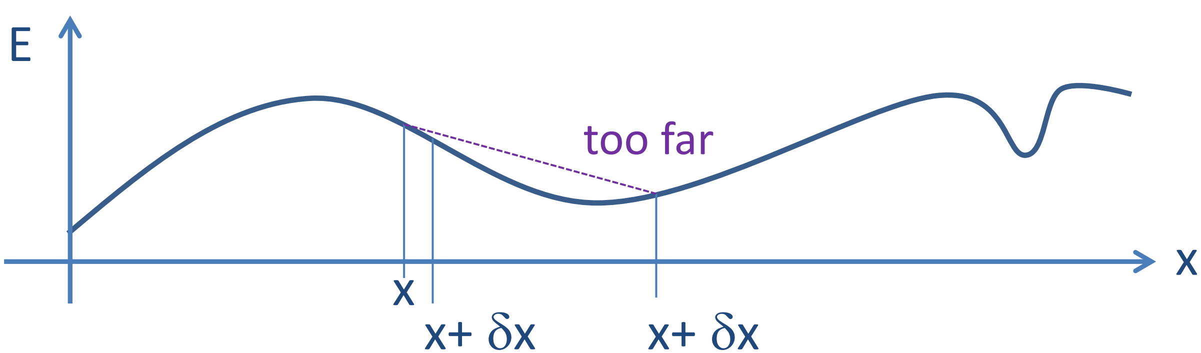
$$E(x + \delta x, y + \delta y, t + \delta t) = E(x,y,t)$$

Taylor Series Expansion

$$E(x,y,t) + \delta x\, E_x + \delta y\, E_y + \delta t\, E_t = E(x,y,t)$$

$$dx/dt\, E_x + dy/dt\, E_y + E_t = 0$$

$$\boxed{u\, E_x + v\, E_y + E_t = 0}$$



Validity depends on spatial frequency of image

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Constant brightness constraint

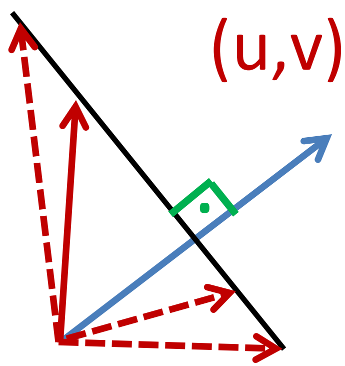
$$u\,E_x + v\,E_y + E_t = 0$$

- How many equations and unknowns per pixel?

  One equation, two unknowns u, v

- Intuitively, what does this constraint mean?

$$(u,v)^T (E_x,E_y) = -E_t$$



Infinite number of possible flows $(u,v)^T$

We don't know where on straight line.

- The component of the flow perpendicular to the intensity gradient (E_x,E_y) (i.e., parallel to the edge) is unknown

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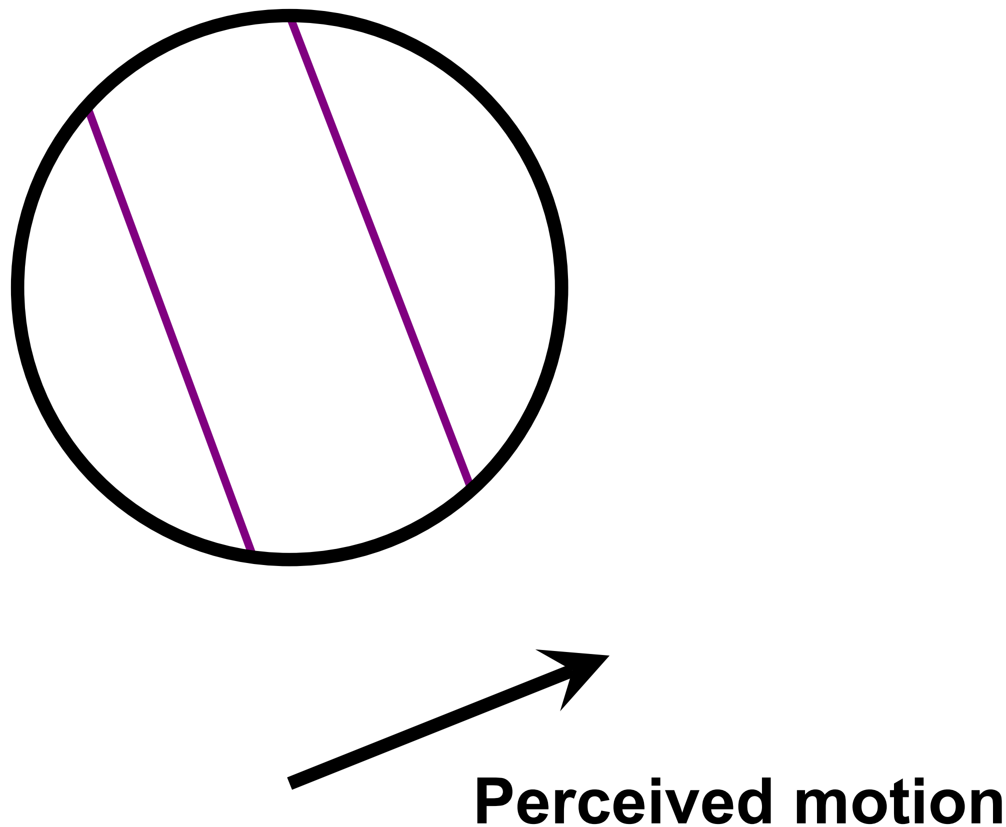
The aperture problem



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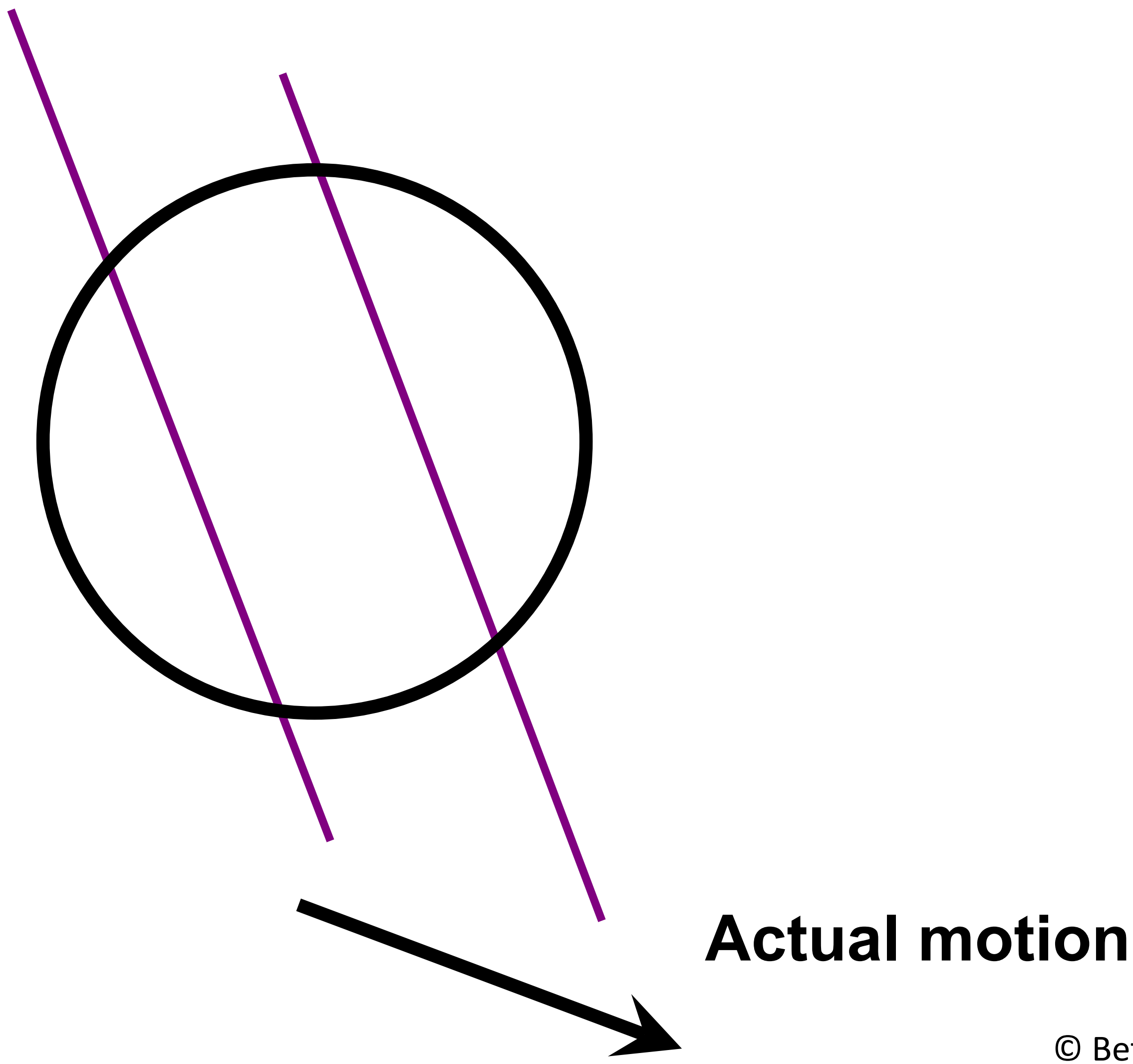
The aperture problem



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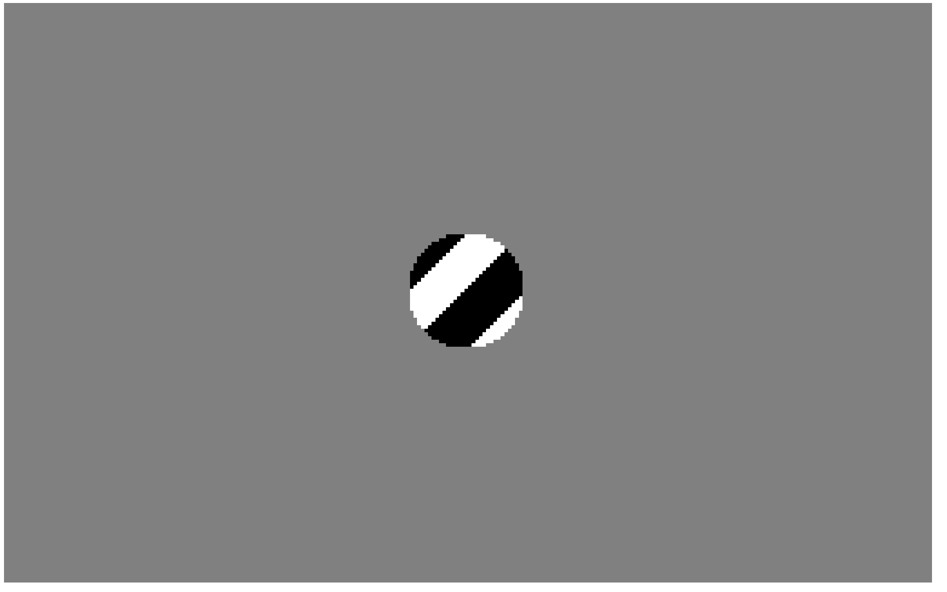
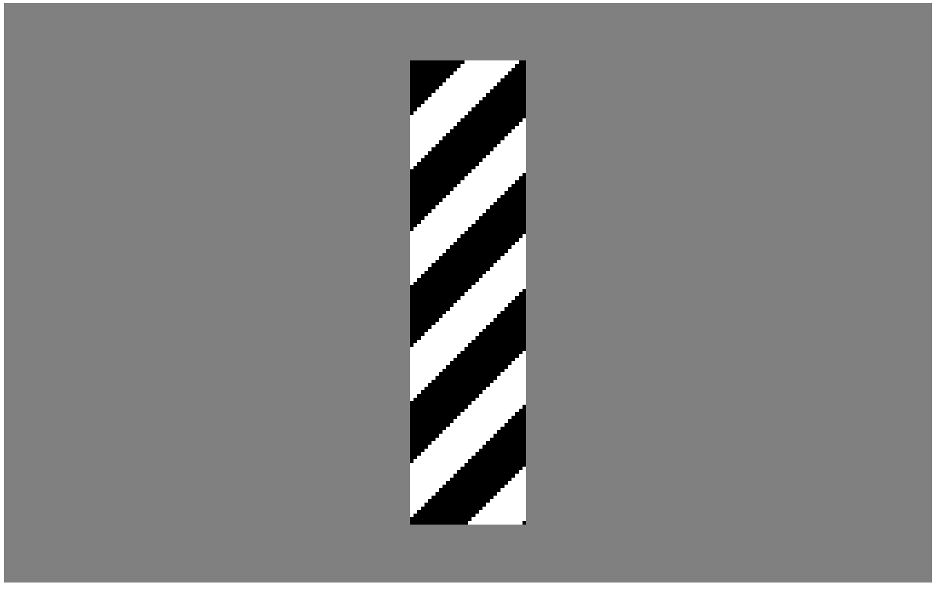
The barber pole illusion



http://en.wikipedia.org/wiki/Barberpole_illusion

The barber pole illusion



http://en.wikipedia.org/wiki/Barberpole_illusion

The barber pole illusion

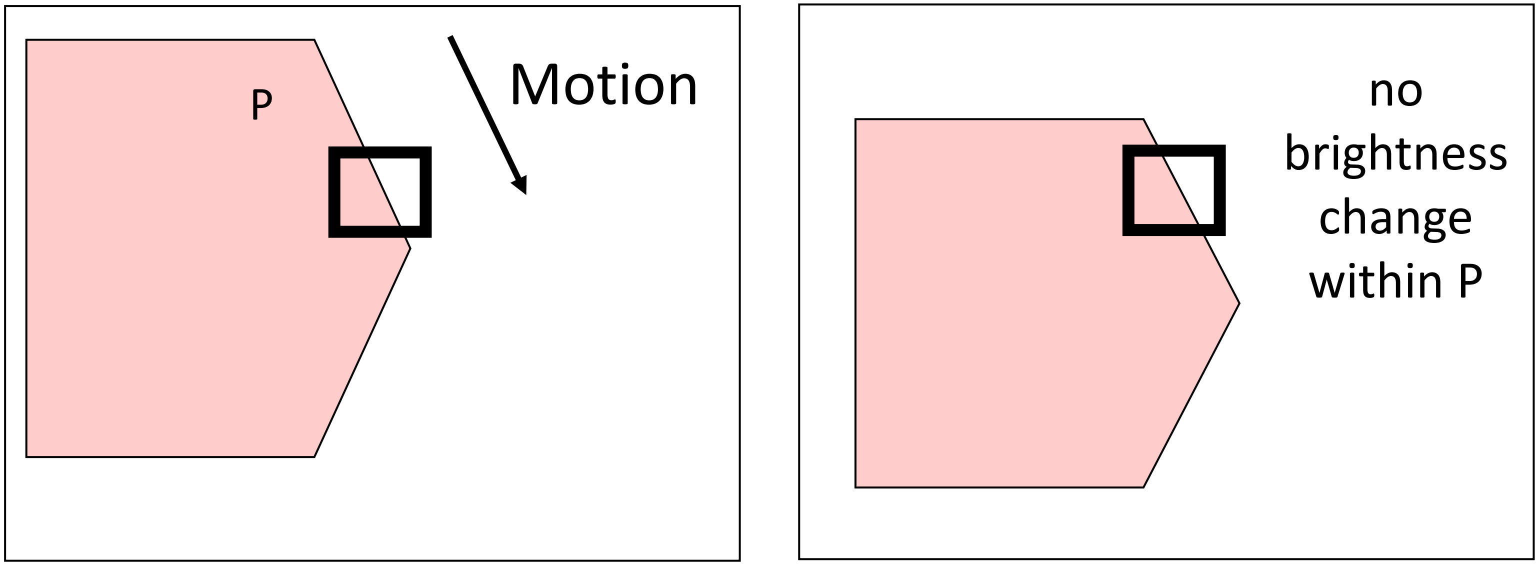
http://en.wikipedia.org/wiki/Barberpole_illusion

Problems with Optical Flow

Aperture Problem

1) Flow perpendicular to brightness gradient

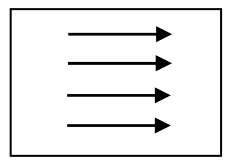
=> Cannot compute u, v



Problems with Optical Flow

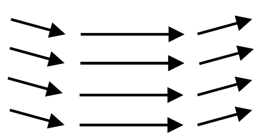
Aperture Problem

2) Only a small portion of flow field is given



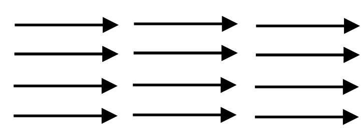
can both represent

rotation



or

translation



Lukas & Kanade Optical Flow Algorithm

- How to get more equations for a pixel?
- **Spatial coherence constraint:** Pretend the pixel's neighbors have the same (u,v) $u\,E_x + v\,E_y + E_t = 0$
- 5x5 window => 25 equations per pixel

$$\begin{pmatrix} E_{x,1} & E_{y,1} \\ E_{x,2} & E_{y,2} \\ \ldots & \\ E_{x,25} & E_{y,25} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} E_{t,1} \\ E_{t,2} \\ \\ E_{t,25} \end{pmatrix}$$

B. Lucas and T. Kanade. An iterative image registration technique with an application to stereo vision. In *Proceedings of the International Joint Conference on Artificial Intelligence*, pp. 674–679, 1981.

Lukas & Kanade Optical Flow Algorithm

$$\begin{pmatrix} E_{x,1} & E_{y,1} \\ E_{x,2} & E_{y,2} \\ \dots & \\ E_{x,25} & E_{y,25} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} E_{t,1} \\ E_{t,2} \\ \\ E_{t,25} \end{pmatrix}$$

$$A \begin{pmatrix} u \\ v \end{pmatrix} = b$$

25x2 2x1 25x1

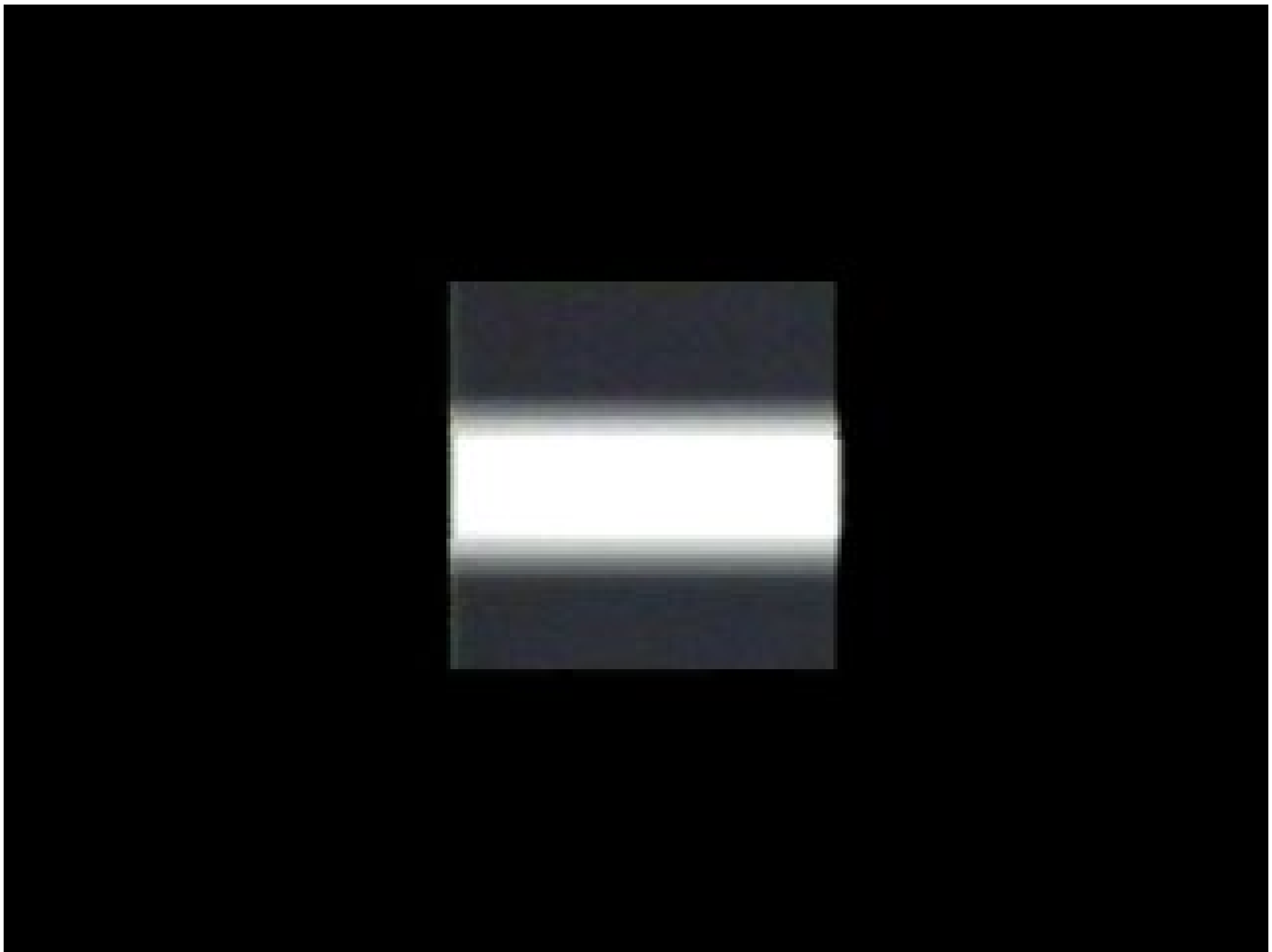
1. When is this system solvable?

2. What if the window contains just a single straight edge?

B. Lucas and T. Kanade. An iterative image registration technique with an application to stereo vision. In *Proceedings of the International Joint Conference on Artificial Intelligence*, pp. 674–679, 1981.

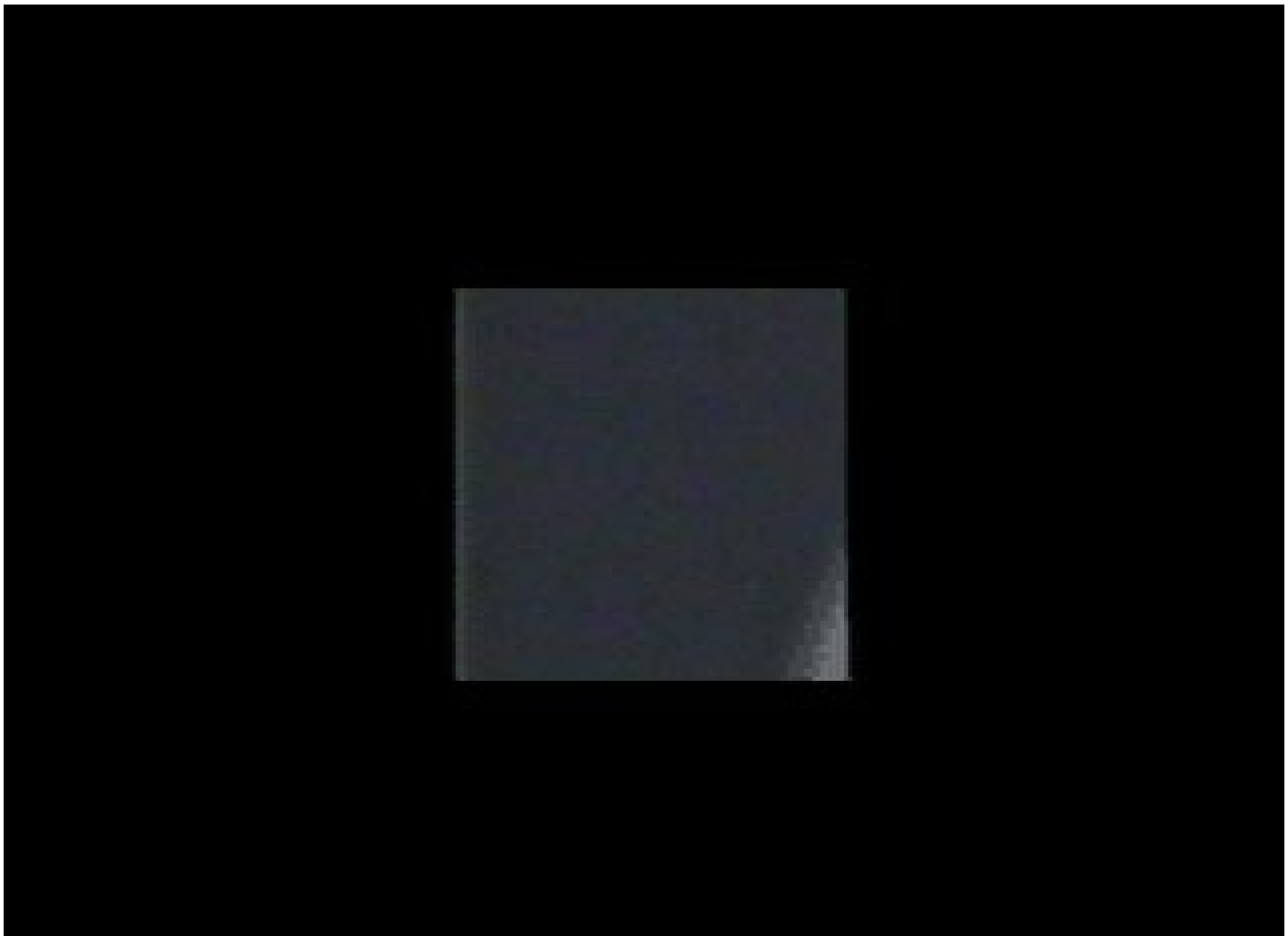
Conditions for solvability

- “Bad” case: Single straight edge



Conditions for solvability

- “Good” case: Smooth change of brightness



Lucas-Kanade Optical Flow Algorithm

Linear least squares problem

$$A \begin{pmatrix} u \\ v \end{pmatrix} = b$$

25x2 2x1 25x1

Solution given by

$$A^T A \begin{pmatrix} u \\ v \end{pmatrix} = A^T b$$

$$\begin{bmatrix} \sum E_x^2 & \sum E_x E_y \\ \sum E_x E_y & \sum E_y^2 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = - \begin{pmatrix} \sum E_x E_t \\ \sum E_y E_t \end{pmatrix}$$

The summations are over all pixels in the window.

Lucas-Kanade Optical Flow Algorithm

$$\underbrace{\begin{bmatrix} \sum E_x^2 & \sum E_x E_y \\ \sum E_x E_y & \sum E_y^2 \end{bmatrix}}_{A^T A} \begin{pmatrix} u \\ v \end{pmatrix} = - \underbrace{\begin{pmatrix} \sum E_x E_t \\ \sum E_y E_t \end{pmatrix}}_{A^T b}$$

- $M = A^T A$ is the "*second moment matrix*"
- Unique solution for flow vector (u,v) ?

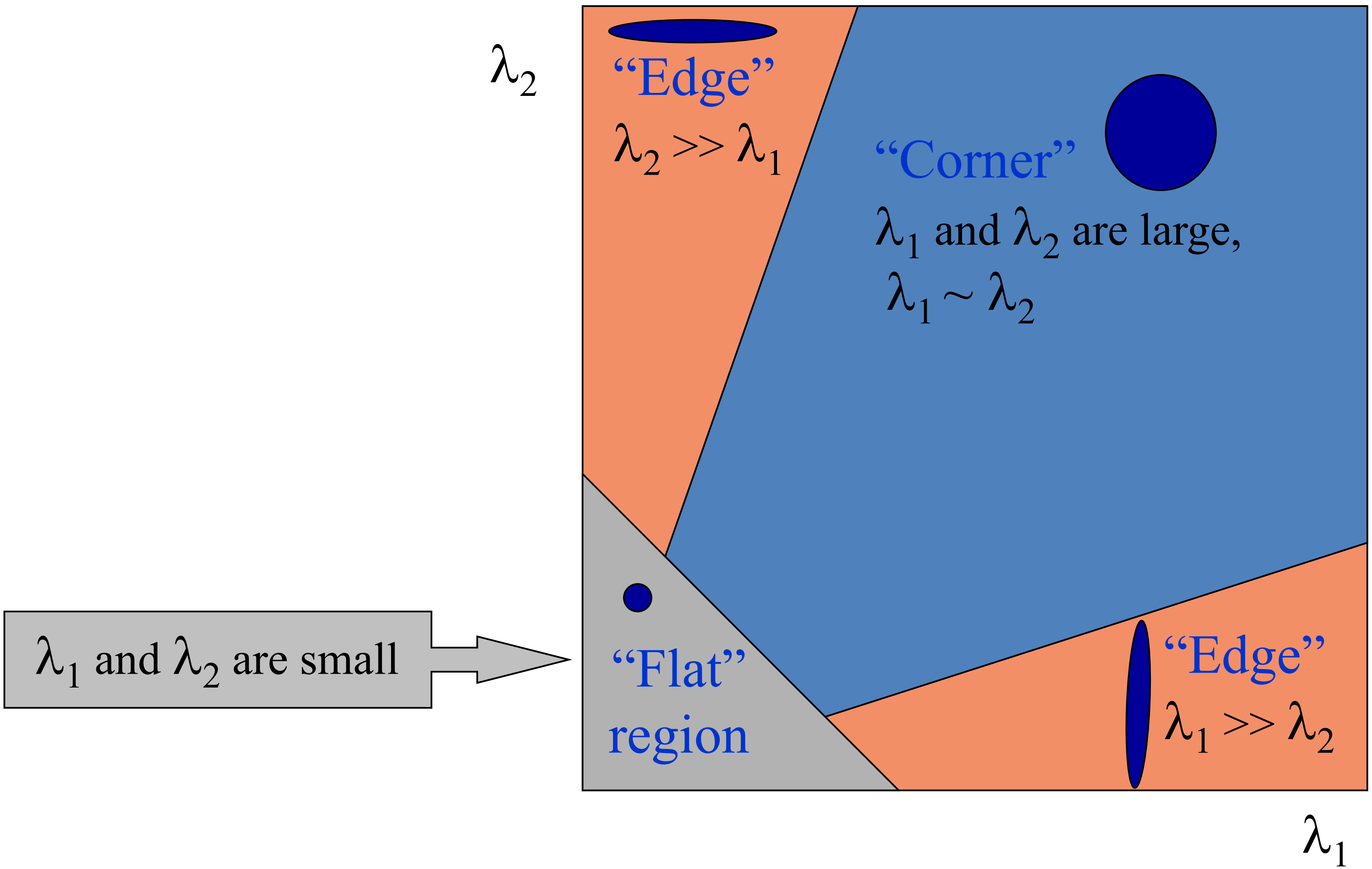
  = eigenvalues of the second moment matrix?

  Eigenvectors and eigenvalues of *M* relate to edge direction and magnitude

  Eigenvector associated with the larger eigenvalue points in the direction of fastest intensity change, and the other eigenvector is orthogonal to it

Interpreting the Eigenvalues

Classification of image points using eigenvalues of the second moment matrix:



Uniform region

– gradients have small magnitude
– small λ_1, small λ_2
– system is ill-conditioned

Edge

– gradients have one dominant direction

– large λ_1, small λ_2

– system is ill-conditioned

High-texture or corner region

– gradients have different directions, large magnitudes
– large λ_1, large λ_2
– system is well-conditioned

Horn and Schunk's Optical Flow Algorithm: CBA & SA

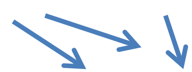
Smoothness Assumption (SA)

Use spatial derivatives of flow: $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$

Magnitude of the flow gradient $= 0$

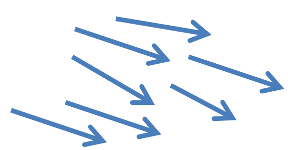
$$(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2$$

Patch size: Small



CBA okay, SA weak

Large



CBA may be violated, SA strong

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Horn and Schunk's Optical Flow Algorithm: CBA & SA

Retrieve flow $(u, v)^T$ by reducing the error in CBA and

Use regularization, weigh errors with a scalar α:

$$\min_{(u,v)} \sum_{patch} (\alpha \ \text{error}_{CBA} + \text{error}_{SA})$$

$$\min_{(u,v)} \sum_{patch} (\alpha \ (uE_x + vE_y + E_t)^2 + (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2)$$

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Horn and Schunk’s Optical Flow Algorithm: CBA & SA

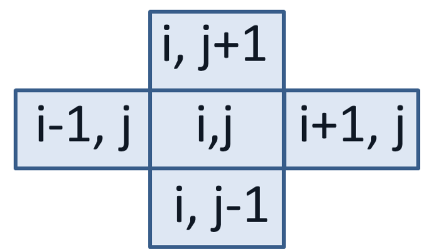
Retrieve flow $(u, v)^T$ by reducing the error in CBA and

Use regularization, weigh errors with a scalar α:

$$\min_{(u,v)} \sum_{patch} (\alpha \ \text{error}_{CBA} + \text{error}_{SA})$$

$$\min_{(u,v)} \sum_{patch} (\alpha \ (uE_x + vE_y + E_t)^2 + (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial v}{\partial y})^2)$$

$$\min_{(u,v)} \sum_{(i,j)\in P} (\alpha \ (u_{i,j}E_x + v_{i,j}E_y + E_t)^2 +$$



$$\frac{1}{4}[(u_{i+1,j} - u_{i,j})^2 + (u_{i,j+1} - u_{i,j})^2 + (v_{i+1,j} - v_{i,j})^2 + (v_{i,j+1} - v_{i,j})^2]$$

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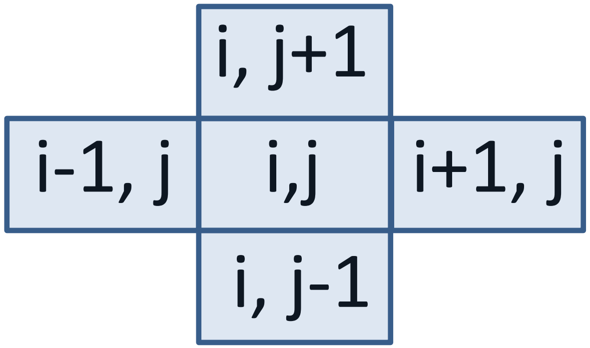
Horn and Schunk's Optical Flow Algorithm: CBA & SA



$$\bar{u} = u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}$$

$$\bar{v} = v_{i-1,j} + v_{i+1,j} + v_{i,j+1} + v_{i,j-1}$$

$$u_{i,j}^{(n+1)} = u_{i,j}^{(n)} - \alpha \frac{E_x \bar{u}_{i,j}^{(n)} + E_y \bar{v}_{i,j}^{(n)} + E_t}{1 + \alpha(E_x^2 + E_y^2)} E_x$$

$$v_{i,j}^{(n+1)} = v_{i,j}^{(n)} - \alpha \frac{E_x \bar{u}_{i,j}^{(n)} + E_y \bar{v}_{i,j}^{(n)} + E_t}{1 + \alpha(E_x^2 + E_y^2)} E_y$$

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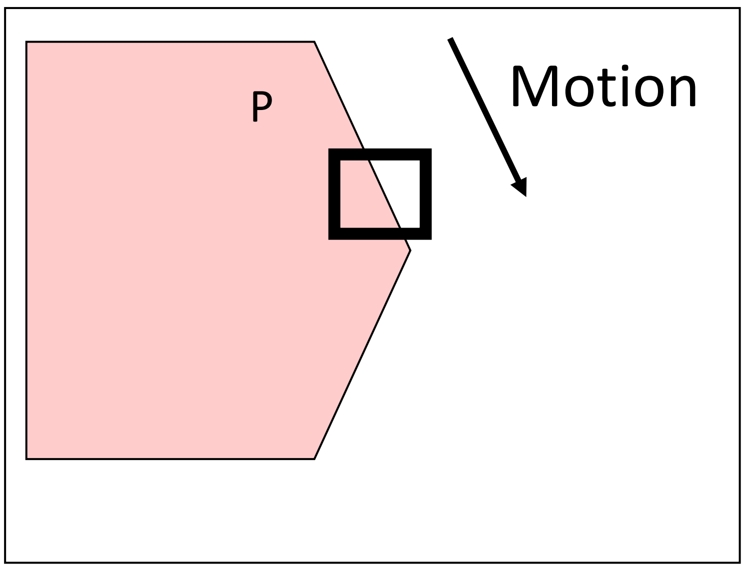
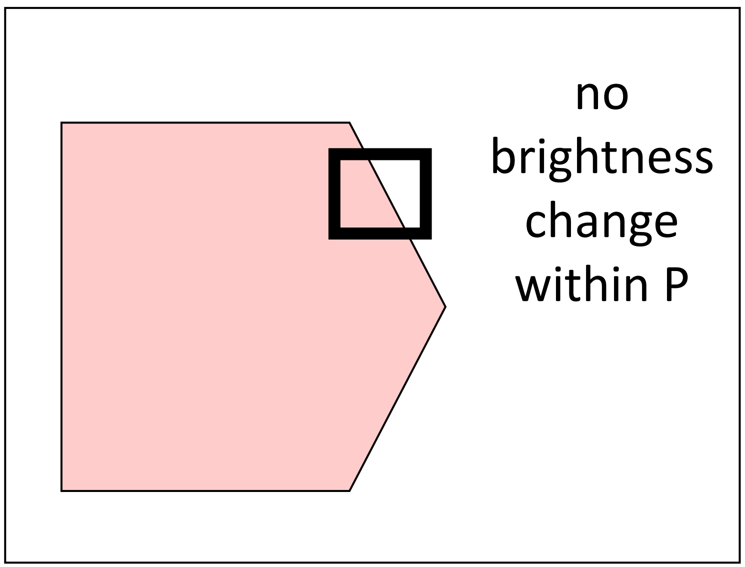
Problems with Optical Flow

"Aperture Problem(s)"

1) Flow perpendicular to brightness gradient

=> Cannot compute u, v

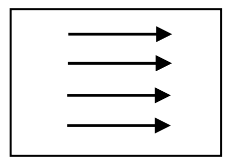




Problems with Optical Flow

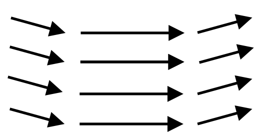
“Aperture Problem(s)”

2) Only a small portion of flow field is given



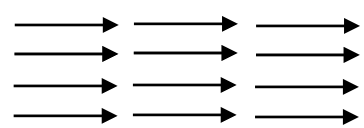
can both represent

rotation



or

translation



What to do when the Optical Flow Algorithm breaks down

- Apparent motion is large (larger than a pixel)
  - Iterative refinement
  - Coarse-to-fine estimation
  - Exhaustive neighborhood search
- A point does not move like its neighbors
  - Motion segmentation
- Constant Brightness Assumption does not hold
  - Exhaustive neighborhood search with normalized correlation

Learning Objectives:

Understand

- Difference between motion flow fields and optical flow fields
- Constand Brightness Assumption (CBA)
- Flow Smoothness Assumption
- Derivation of CBA using calculus
- 2 versions of the “Aperture Problem”
- Lucas-Kanade and Horn-Schunk algorithms

Be able to draw translational & rotational flow fields