CS 332 - Homework 3

Due: Thursday, March 2 at midnight
Note: We will be pretty lenient about late submissions of this Homework, but HW 3 should be doable in a day or two. Don’t waste your whole spring break on it.

Reading: Chapter 4, section 2, pages 201 - 210, and Chapter 5, pages 215-226.

PROBLEMS:

The first 2 problems have to do with closure properties of decidable and recognizable sets. We have seen this in class when we showed that the decidable sets are closed under complement (meaning the complement of a decidable set is decidable)

You can find the first two problems below as part of problems 3.15 and 3.16 in our textbook. And you can find how to prove closure under union by looking at the answers given by Sipser on page 191.

1. Show that the collection of decidable languages is closed under the operation of
   (a). intersection and
   (b). set subtraction.

2. The direct product of two sets T and U is the set of all ordered pairs, \( T \times U = \{ (x,y) \mid x \in T \text{ and } y \in U \} \).

   Show that the collection of recognizable languages is closed under the operation of
   (a). intersection and
   (b). direct product.

3. (This problem is from Sipser’s book, page 188.)
   (i). Explain why the following is not a description of a legitimate Turing machine.
   The input to the Turing machine is a polynomial \( p \) with integer coefficients, over two variables \( x_1 \) and \( x_2 \).
   1. Try all possible settings of \( x_1, x_2 \) to integer values.
   2. Evaluate \( p(x_1, x_2) \) on all of the integers.
   3. If any of these polynomial with integer inputs evaluate to 0, accept; otherwise, reject.

   (ii). Now remove the words “otherwise, reject” from line 3 above.
   The TM described in part (i) above is now a legitimate Turing machine T, and T shows that the set of two-variable polynomials described above is recognized by T. Briefly explain why this is true.
4. Consider an enumerator $E$ which enumerates an infinite set $S$ of binary strings. Recall that such a set is recognizable (see theorem 3.21, page 181 in our textbook).

Now construct an infinite subset $T = \{t_1, t_2, t_3, \ldots\}$ of $S$ using the following Algorithm A

**Algorithm A:**

1. Let $t_1$ = the first binary string that $E$ enumerates on its print tape.

2. For $i = 2, 3, 4, \ldots$, let $t_i$ = the next string enumerated which is longer than $t_{i-1}$.

(i). Prove that $T$ is a subset which enumerates an infinite subset of $S$

(ii). Prove that $T$ is decidable.

Note that (i) and (ii) together imply that any infinite enumerable set $S$ has an infinite decidable subset.

And by 3.21, this holds for any infinite recognizable set $S$.

Addendum to HW 4: The way I wrote up problem 4(i) on homework 3 is pretty confusing.

I will try to change this in a few minutes on the HW 3 on the CS 332 Home page and also on Pizza. But let me try to write more clearly what $S$ and $T$ are, that is how they are constructed.

In this case we construct 2 sets $S$ and $T$ together in one algorithm. The algorithm uses both an enumerator TM which prints the set $S$ with its printer. It is pictured on page 180 of our textbook. Nothing is new here at all.

Also on page 180 you can see that the enumerator TM also has a work tape. In this case the enumerator TM lists on its work tape a set $T$ which contains some of the elements of $S$ as the TM prints them. Other strings which are enumerated in $S$ may not get put into $T$. So clearly $T$ is a subset of $S$.

The algorithm A given in problem 4(i) of HW 3 tells you which elements of $S$ are put into $T$ as $S$ is being enumerated. Specifically, assuming that $S$ is infinite, it tells you how to define $T$ the infinite subset of $S$ is defined.

Your job is to answer part (i) and part(ii) of HW 3. Each of these parts are true statements and you have to explain why they are true because of how we construct $T$ from $S$. (That’s it.)