BU CLA CS 530: Analysis of Algorithms

Spring 1999

Problem Set 3

Due Thursday Feburary 4, by 5:00 pm, in my mailbox in the main office of the CS department. Late submissions not accepted.

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1. Exercise 16.3-5, page 319, in [CLR].
   Hint: Sort the input sequence into a second sequence.

2. Exercise 16.3-6, page 319, in [CLR].
   Hint: Use the hint given in the book. Also note that the trick of the preceding exercise (i.e., sort the input sequence into a second sequence) will not suffice, even if you sort the input sequence in time $\mathcal{O}(n \log n)$.

3. Exercise 16.4-2, page 324, in [CLR].
   Hint: Don't be misled. This is really easy, if you can view the problem in a certain way.

4. Exercise 31.2-3, page 744 in [CLR]. (The definition of "little-oh" is given at the bottom of page 29 in [CLR]. The simpler definition to work with is given at the top of page 30, namely, we write $f(n) = o(g(n))$ iff $\lim_{n\to\infty}(f(n)/g(n)) = 0$.)
   Hint: Set up an appropriate recurrence equation, and consider how you can solve it using the Master theorem on page 62 in [CLR].

5. Exercise 31.5-1, page 765 in [CLR].
   Hint: This is easy, if you look at the problem the right way.

Further Comments and Hints for Exercise 16.3-6 in [CLR]

Given a sequence X of $n \geqslant 1$ numbers, say $X = \langle x_1, \ldots, x_n \rangle$, a *monotonically increasing subsequence* (MIS) of X is a sequence $S = \langle x_{i_1}, \ldots, x_{i_k} \rangle$ satisfying two conditions:

1. $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$,
2. $x_{i_1} < x_{i_2} < \ldots < x_{i_k}$.

The sequence S above is a *longest* MIS of X if for every MIS T of X, say $T = \langle x_{j_1}, \ldots, x_{j_\ell} \rangle$, it also holds that:

3. $k \geqslant \ell$.

For example, if $X = \langle 2, 3, 1, 8, 9, 2, 5, 6, 4 \rangle$ then

- $\langle 2, 8, 9 \rangle$ and $\langle 3, 5, 6 \rangle$ are MIS's of X but not longest ones, among many others.
- $\langle 2, 3, 8, 9 \rangle$, $\langle 2, 3, 5, 6 \rangle$ and $\langle 1, 2, 5, 6 \rangle$ are longest MIS's of X, and there are only 3 of them.

As in Section 16.3 in [CLR], let $X_k = \langle x_1, \ldots, x_k \rangle$ for every $k \in \{1, 2, \ldots, n\}$. Define:

$c[k]$ = the length of a longest MIS of X_k.

We want an efficient algorithm, which runs in time $\mathcal{O}(n \log n)$, to compute the n entries in $c[1..n]$. We do this by a small detour to investigate a related measure — as explained next.

Note that different MIS's of X may end with different values. For every $k \in \{1, 2, \ldots, n\}$, it is convenient to introduce an array $b_k[1..k]$ with k entries such that, for every $\ell \in \{1, 2, \ldots, k\}$:

$b_k[\ell]$ = the minimum value ending a MIS of X_k of length ℓ.

If there is no MIS of X_k of length ℓ, we set $b_k[\ell] = \infty$, where ∞ is a "very large number". For the example above, if we choose $k = 7$ an consider $X_7 = \langle 2, 3, 1, 8, 9, 2, 5 \rangle$ then, by inspection,

$$c[7] = 4 \quad \text{and} \quad b_7[1] = 1,\ b_7[2] = 2,\ b_7[3] = 5,\ b_7[4] = 9 \quad \text{and} \quad b_7[5] = b_7[6] = b_7[7] = \infty \ .$$

A key observation is that array $b_k[1..k]$ is ordered, i.e., for some $m \in \{1, 2, \ldots, k\}$ we always have:

$$(\dagger)\ \ b_k[1] < b_k[2] < \cdots < b_k[m] < b_k[m+1] = \cdots = b_k[k] = \infty \ .$$

This is so because the minimum value ending a MIS of length ℓ must be smaller than the minimum value ending a MIS of length $\ell + 1$. Convince yourself of the truth of $(\dagger)$ before proceeding further.

Instead of a 1-dimensional array $b_k[1..k]$, one for each $k \in \{1, 2, \ldots, n\}$, we can define a single $n \times n$ matrix $b[1..n, 1..n]$ where all the entries $\neq \infty$ are on the diagonal or below it. Put differently,

$$b[k, 1] = b_k[1],\ \ldots\ ,\ b[k, k] = b_k[k] \quad \text{and} \quad b[k, k+1] = \cdots = b[k, n] = \infty \ .$$

Before computing $c[1..n]$, we compute the entries in $b[1..n, 1..n]$ in row-major order. The first row $b[1, 1..n]$ is defined by:

$(*)$ $b[1,1] = x_1$ and $b[1,2] = \cdots = b[1,n] = \infty$.

Assume that row k, for some $k \geqslant 1$, is already determined. We obtain row $k+1$ from row k by making a single substitution, namely, we substitute x_{k+1} for the first entry in $b[k, 1 \,..\, n]$ from left to right which is not smaller than x_{k+1}. More specifically, if ℓ is the least index in $\{1, 2, \ldots, n\}$ such that $x_{k+1} \leqslant b[k, \ell]$, then $b[k+1, 1 \,..\, n]$ is defined by:

$$
\begin{aligned}
(**)\quad & b[k+1,1] = b[k,1], \ \ldots\ , \ b[k+1,\ell-1] = b[k,\ell-1], \\
& b[k+1,\ell] = x_{k+1}, \\
& b[k+1,\ell+1] = b[k,\ell+1], \ \ldots\ , \ b[k+1,n] = b[k,n] \ .
\end{aligned}
$$

Convince yourself that $(*)$ and $(**)$ together define $b[1 \,..\, n, 1 \,..\, n]$ properly according to the desired specification, i.e., $b[k, \ell]$ is the minimum value ending a MIS of X_k of length ℓ. We are ready to write a recurrence for $c[k]$:

$$
(\ddagger)\quad c[k+1] = \begin{cases} 1 & \text{if } k = 0, \\ 1 + c[k] & \text{if } k \geqslant 1 \text{ and } x_{k+1} > \max\{b[k,\ell] \mid 1 \leqslant \ell \leqslant n \text{ and } b[k,\ell] \neq \infty\}, \\ c[k] & \text{if } k \geqslant 1 \text{ and } x_{k+1} \leqslant \max\{b[k,\ell] \mid 1 \leqslant \ell \leqslant n \text{ and } b[k,\ell] \neq \infty\}. \end{cases}
$$

This recurrence says that $c[k+1] = 1 + c[k]$ if and only if x_{k+1} is larger than the minimum value ending a longest MIS of X_k. Convince yourself of the truth of $(\ddagger)$.

I leave it to you to complete the analysis above. More precisely, if we measure time by the number of comparisons carried out, do the following:

(a) Show how the matrix $b[1 \,..\, n, 1 \,..\, n]$ can be computed in time $\mathcal{O}(n \log n)$.

(b) With $b[1 \,..\, n, 1 \,..\, n]$ as input, show how the array $c[1 \,..\, n]$ can be computed in time $\mathcal{O}(n \log n)$.

(c) With $b[1 \,..\, n, 1 \,..\, n]$ as input, design a $\mathcal{O}(n)$ procedure that returns a longest MIS of X. Specifically, the procedure returns an array $d[1 \,..\, n]$ such that if

$$
d[1] = i_1, \ d[2] = i_2, \ \ldots\ , \ d[k] = i_k \quad \text{and} \quad d[k+1] = \cdots = d[n] = \infty
$$

then $\langle x_{i_1}, \ldots, x_{i_k} \rangle$ is a longest MIS of X.

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