BU CLA CS 530: Analysis of Algorithms
Spring 1999

Selected Solutions for Problem Set 10

Problem 1. (Exercise 36.2-8, page 929, in [CLR])
Let φ range over boolean formulas. Then:

$$\text{TAUTOLOGY} = \{\varphi \mid \varphi \text{ evaluates to 1 for every truth assignment}\}$$

Recall that a "truth assignment" is an assignment of 1 (true) and of 0 (false) to variables. The complement of TAUTOLOGY is:

$$\text{FALSIFIABLE} = \{\varphi \mid \varphi \text{ evaluates to 0 for some truth assignment}\}$$

It suffices to show that FALSIFIABLE is NP. This is shown just as SAT is shown to be NP (details omitted).[1] In fact we can show more (not required in the exercise) : FALSIFIABLE is NP-hard and therefore NP-complete. To show that FALSIFIABLE is NP-hard, we prove the reduction:

$$\text{SAT} \leqslant_P \text{FALSIFIABLE}$$

For an input instance φ of SAT, define an input instance $\varphi^\star$ of FALSIFIABLE such that truth assignment t satisfies φ iff truth assignment $\overline{t}$ falsifies $\varphi^\star$. $\overline{t}$ is the complement of t, i.e. if $t(x) = 0$ (respectively 1) then $\overline{t}(x) = 1$ (respectively 0). The desired $\varphi^\star$ is obtained from φ by changing every $\vee$ to $\wedge$. The proof that the reduction is correct is by induction on the definition of boolean formulas φ (details omitted).

Problem 2.
Let $G = (V, E)$ be an undirected graph. Because each edge in E contributes a count of 1 to the degree of each of the vertices with which it is incident, the sum of the degrees of the vertices is exactly $2 \cdot |E|$, an even number. Hence, there is an even number of vertices in G that have odd degrees.

Let now $\langle G, k\rangle$ be an input instance of VERTEX-COVER, where $G = (V, E)$ as above. Let U be the subset of vertices with odd degrees in G; $|U|$ is even. Construct a new instance $\langle G', k+2\rangle$ of VERTEX-COVER, where $G' = (V', E')$ with $V' = V \cup \{x, y, z\}$ (three new vertices) and

$$E' \;=\; E \;\cup\; \{\ \{x,y\},\ \{x,z\},\ \{y,z\}\ \} \;\cup\; \{\ \{x,v\} \mid v \in U\ \}$$

In words, we make a triangle with the three new vertices, and then connect one of them (say x) to all the vertices in U. The degree of every vertex in V' is even. Because a vertex cover for a triangle is of (minimum) size 2, it is clear that G' has a vertex cover of size $k+2$ iff G has a vertex cover of size k.

Problem 3.
Let $\langle G, k\rangle$ be an instance of VERTEX-COVER. We polynomially transform $\langle G, k\rangle$ into an instance $\langle G', k'\rangle$ of CLIQUE such that G has a vertex cover of size $\leqslant k$ iff G' has a clique of size $\geqslant k'$. If $G = (V, E)$ then the desired $G' = (V', E')$ is defined by:

[1] If you can't handle the case of arbitrary formulas, try the case of formulas in DNF (disjunctive normal form) first. *Warning*: FALSIFIABLE restricted to formulas in CNF is polynomial-time solvable, just as SAT restricted to formulas in DNF is polynomial-time solvable — see Exercise 36.4-5, page 946, in [CLR].

$$V' = V \text{ and } E' = (V \times V) - E$$

There is no loss of generality in assuming that $k < n = |V|$. Set $k' = n - k$. The transformation from $\langle G, k\rangle$ to $\langle G', k'\rangle$ is polynomial time.

Consider now a vertex cover $U \subset V$ for G of size $\leqslant k$ and let $T = V - U$. For all $v, w \in V$, if $(u, v) \in E$ then $v \in U$ or $w \in U$ or both (because U is a vertex cover). The contrapositive of this implication is : For all $v, w \in V$, if $v \notin U$ and $w \notin U$ then $(u, v) \in E'$. In other words, $V - U = T$ is a clique for G' of size $\geqslant n - k = k'$. (This argument really reproduces the end of the proof of Theorem 36.12 in [CLR], p. 950.)

Conversely, if T is a clique for G' of size $\geqslant k'$, the set $U = V - T$ is a vertex cover for G of size $\leqslant k$. (Details omitted, as the argument is already in the proof of Theorem 36.12).

Problem 4.

(a) (Exercise 36.2-6, page 929, in [CLR]) We have to show that HAM-PATH is in NP. Given a graph $G = (V, E)$ together with 2 distinct vertices u and v in V, our certificate is a sequence of vertices that make up a hamiltonian path from u to v. The verification algorithm checks that this sequence is of length $|V|$, that it starts at u and ends at v, and that every vertex in the sequence occurs exactly once and is connected to the vertex preceding it in the sequence. This verification can be done in $\mathcal{O}(n^2)$ steps (why?), which is polynomial.

(b) (Exercise 36.5-5, page 961, in [CLR]) Based on part (a) above, it suffices to show that HAM-PATH is NP-hard. From part (c) below, we also know that HAM-PATH$'$ is NP-complete. To show therefore that HAM-PATH is NP-hard, it suffices to show that HAM-PATH$'$ $\leq_P$ HAM-PATH. (This is not the only way of proving HAM-PATH is NP-hard.)

An input instance of HAM-PATH$'$ is an undirected graph $G = (V, E)$. For each pair of distinct vertices u and v in V, we obtain an input instance $\langle G, u, v\rangle$ of HAM-PATH. If $|V| = n$, there are $\binom{n}{2} = n\cdot(n-1)/2$ such input instances of HAM-PATH. The input instance of HAM-PATH$'$ produces a "yes" answer if and only if one of the $n \cdot (n - 1)/2$ input instances of HAM-PATH produces a "yes" answer. (I have used here the more general notion of "polynomial-time reduction" presented in lecture.)

(c) This was done in lecture, on Tuesday March 30, based on splitting an arbitrary node $v \in V$ into v' and v''. If v has degree 2, this splitting operation needs to be done only once. If v has degree $k \geqslant 2$, the splitting operation has to be done $\binom{k}{2} = k \cdot (k - 1)/2$, which is of course a polynomial in the size of the input graph. I omit writing the details of a complete solution.

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