BU CLA CS 530: Analysis of Algorithms
Spring 1999

Selected Solutions for Problem Set 13

Problem 1.

The answer is *no*, VCIS is not NP-hard, and there is a polynomial-time (actually linear-time) algorithm to solve it. Let us start with some more general facts about (undirected) graphs, not all needed for this exercise but all worth remembering. Given an undirected graph $G = (V, E)$, we have the following equivalences:

$$\begin{aligned} G \text{ is bipartite} &\iff G \text{ is 2-colorable} \\ &\iff G \text{ has no odd-length cycles} \\ &\iff G \text{ has a vertex cover which is an independent set} \end{aligned}$$

These equivalences are easy to prove (left to you). We can easily check whether G is bipartite or 2-colorable, using DFS or BFS (Chapter 23 in [CLR]) in time $\mathcal{O}(V + E)$. We omit the details of the pseudo-code. But we are not done yet, because if G is bipartite (i.e. there is a subset of V which is both a vertex cover and an independent set) we are asked to return one of minimum size. If G is bipartite, let L and R be respectively its "left" and "right" vertices which, in addition, are not isolated. (A vertex v is isolated if no edge is incident on v.) If $|L| \leq |R|$ then return L else return R.

Problem 2.

Call TSP$'$ the restriction of TSP satisfying the triangle inequality. The argument showing that TSP is in NP applies to TSP$'$. It remains to show that TSP$'$ is NP-hard. We prove this by the reduction HAM-CYCLE $\leq_{\mathrm{P}}$ TSP$'$. Given an instance $G = (V, E)$ of HAM-CYCLE, with $|V| = n$, we construct an instance of TSP$'$ consisting of a graph $G' = (V, V \times V)$ and a cost function $c' : V \times V \to \{1, 2\}$ satisfying the triangle inequality:

$$c'(u, v) = \begin{cases} 1, & \text{if } (u, v) \in E, \\ 2, & \text{if } (u, v) \notin E. \end{cases}$$

It is clear that c' satisfies the triangle inequality and, moreover, can be defined in time polynomial in the size of the given G (why?). A tour in G' of length n must contain only edges of cost 1 and, therefore, corresponds directly to a hamiltonian cycle in G, i.e. G has a hamiltonian cycle iff G' has a tour of length n.

Another simple proof for the NP-hardness of TSP$'$ follows from the reduction TSP $\leq_{\mathrm{P}}$ TSP$'$. For this reduction, define a cost function $c' : V \times V \to \mathbf{Z}$ from the given cost function c by choosing a positive integer k and setting: $c'(u, v) = c(u, v) + k$, for all $u, v \in V$. The trick is to choose k large enough in order to satisfy the triangle inequality — choosing $k \geq 2 \cdot \max\{|c(u, v)| : u, v \in V\}$ for example will do (details left to you).

Problem 3. (Exercise 37.2-1, page 973, in [CLR])

The first part of this exercise follows from the reduction TSP $\leq_{\mathrm{P}}$ TSP$'$, given in the solution of the preceding problem, which had the property that a given instance G of TSP and the corresponding instance G' of TSP$'$ have the same set of optimal tours.

To see why this result does not contradict Theorem 37.3 in [CLR], page 972, consider how we defined the cost function c' from the given cost function c: The same constant $k \geq 1$ is added to the cost of every edge, so that G has an optimal tour of cost C^* iff G' has an optimal tour of cost $C^* + n \cdot k$, where $n = |V|$. Now, if we find a suboptimal tour in G' with cost C' and ratio bound ≤ 2, i.e. $C' \leq 2 \cdot C^* + 2 \cdot n \cdot k$, then the corresponding suboptimal tour in G has cost $C \leq 2 \cdot C^* + n \cdot k$ (because $C = C' - n \cdot k$) and, therefore, ratio bound $C/C^* \leq 2 + (n \cdot k/C^*)$ — which can be much larger than 2.

For a concrete example, consider the instance G of TSP in Figure 36.19 in [CLR], page 959: It does not satisfy the triangle inequality. The instance G' of TSP$'$ is obtained by adding any $k \geq 2 \cdot 5$ to the cost of every edge in G (5 is the maximum edge-cost in this example). Choose $k = 10$. An optimal tour in G', say u — x — v — w — u, has cost 47, and the corresponding optimal tour in G has cost 7. A suboptimal tour in G' is u — v — w — x — u which has cost 54 and ratio bound $(54/47) = 1.149$, while the corresponding suboptimal tour in G has cost 14 and ratio bound $(14/7) = 2$. You can make the ratio bound of the suboptimal tour in G worse than 2 by changing the cost of edge u — v from 4 to 5.

Problem 4. (Exercise 37.3-2, page 978, in [CLR])

[CLR] restricts the notion of "NP-completeness" to decision problems, because "polynomial-time verification" is only defined for decision problems (Section 36.2). "NP-hardness", on the other hand, applies to both decision and optimization problems. On page 974 in [CLR], **set-cover** is defined as an optimization problem. For consistency, therefore, this exercise should only ask for the NP-hardness of **set-cover**. We show it by reduction from **vertex-cover** which we take, for definiteness, in its optimization version.

Given an instance of **vertex-cover**, some undirected graph $G = (V, E)$, we construct an instance $\langle X, \mathcal{F} \rangle$ of **set-cover**. Let $V = \{1, \ldots, n\}$. We set $X = E$ and $\mathcal{F} = \{S_1, \ldots, S_n\}$, where for every $i \in \{1, \ldots, n\}$:

$$S_i \;=\; \{\; e \in E \mid \text{edge } e \text{ is incident on vertex } i \;\}$$

$\langle X, \mathcal{F} \rangle$ is constructed from G in polynomial time (why?). Moreover, $\{i_1, \ldots, i_k\} \subseteq V$ is a vertex cover for G iff $S_{i_1}, \ldots, S_{i_k}$ is a set cover for $\langle X, \mathcal{F} \rangle$.

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