CS520, HANDOUT 02, 7 September 2006

Language of Arithmetical Expressions

Reading: Chapter 3 in [Pierce]

Syntax

t	::=		*terms*
		v	*value*
		if t then t else t	*conditional*
		succ t	*successor*
		pred t	*predecessor*
		iszero t	*zero test*
v	::=		*values*
		true	*true value*
		false	*false value*
		nv	*numeric value*
nv	::=		*numeric values*
		0	*zero value*
		succ nv	*successor value*

Different Approaches to Semantics [Pierce, Sect 3.4]

- Operational Semantics,
  using abstract machines and transition rules.
- Denotational Semantics,
  using interpretation functions and domain theory.
- Axiomatic Semantics,
  using (weakest) pre-conditions, (strongest) post-conditions and loop invariants.

Small-Step Operational Semantics [Pierce, Sect 3.5]

$$\frac{}{\texttt{if true then } t_2 \texttt{ else } t_3 \longrightarrow t_2} \quad \text{(E-IfTrue)}$$

$$\frac{}{\texttt{if false then } t_2 \texttt{ else } t_3 \longrightarrow t_3} \quad \text{(E-IfFalse)}$$

$$\frac{t_1 \longrightarrow t_1'}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \longrightarrow \texttt{if } t_1' \texttt{ then } t_2 \texttt{ else } t_3} \quad \text{(E-If)}$$

$$\frac{t \longrightarrow t'}{\texttt{succ } t \longrightarrow \texttt{succ } t'} \quad \text{(E-Succ)}$$

$$\frac{}{\texttt{pred } 0 \longrightarrow 0} \quad \text{(E-PredZero)}$$

$$\frac{}{\texttt{pred (succ } nv\texttt{)} \longrightarrow nv} \quad \text{(E-PredSucc)}$$

$$\frac{t \longrightarrow t'}{\texttt{pred } t \longrightarrow \texttt{pred } t'} \quad \text{(E-Pred)}$$

Small-Step Operational Semantics (continued)

$$\frac{}{\texttt{iszero } 0 \rightarrow \texttt{true}} \quad \text{(E-IsZeroZero)}$$

$$\frac{}{\texttt{iszero (succ } nv\texttt{)} \rightarrow \texttt{false}} \quad \text{(E-IsZeroSucc)}$$

$$\frac{t \rightarrow t'}{\texttt{iszero } t \rightarrow \texttt{iszero } t'} \quad \text{(E-IsZero)}$$

Theorem [Pierce, 3.5.4].
Determinacy of one-step evaluation:
If $t \rightarrow t'$ and $t \rightarrow t''$, then $t' = t''$.

- Determinacy may or may not hold.

Definition [Pierce, 3.5.6]. **Terms in normal form**.

Theorem [Pierce, 3.5.7]. **Every value is in normal form**.

Theorem [Pierce, 3.5.8].
Every term in normal form is a value – or is it ???

- Not quite, if we mix boolean and numerical values. There are terms in normal form that are not values.

Definition [Pierce, 3.5.9]. **Multi-step evaluation**:
$\rightarrow^*$ is the reflexive, transitive closure of $\rightarrow$.

Theorem [Pierce, 3.5.11]. **Uniqueness of normal forms**:
If $t \rightarrow^* t'$ and $t \rightarrow^* t''$ where t' and t'' are in normal form, then $t' = t''$.

Theorem [Pierce, 3.5.12]. **Termination of evaluation**:
For every t there is a t' in normal form such that $t \rightarrow^* t'$.

Definition [Pierce, 3.5.15].
Stuck terms, i.e., "run-time errors":
t is stuck if it is in normal form but not a value.

Big-Step Operational Semantics [Pierce, Ex. 3.5.17]

$$\frac{}{v \Downarrow v} \quad \text{(B-Value)}$$

$$\frac{t_1 \Downarrow \texttt{true} \qquad t_2 \Downarrow v}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \Downarrow v} \quad \text{(B-IfTrue)}$$

$$\frac{t_1 \Downarrow \texttt{false} \qquad t_3 \Downarrow v}{\texttt{if } t_1 \texttt{ then } t_2 \texttt{ else } t_3 \Downarrow v} \quad \text{(B-IfFalse)}$$

$$\frac{t \Downarrow nv}{\texttt{succ } t \Downarrow \texttt{succ } v} \quad \text{(B-Succ)}$$

$$\frac{t \Downarrow 0}{\texttt{pred } t \Downarrow 0} \quad \text{(B-PredZero)}$$

$$\frac{t \Downarrow \texttt{succ } nv}{\texttt{pred } t \Downarrow nv} \quad \text{(B-PredSucc)}$$

Big-Step Operational Semantics (continued)

$$\frac{t \Downarrow \mathtt{0}}{\mathtt{iszero}\ t \Downarrow \mathtt{true}} \quad \text{(B-IsZeroZero)}$$

$$\frac{t \Downarrow \mathtt{succ}\ nv}{\mathtt{iszero}\ t \Downarrow \mathtt{false}} \quad \text{(B-IsZeroSucc)}$$

Theorem [Pierce, 3.5.17].
Equivalence of small-step and big-step semantics:
For every term t and value v, it holds that $t \rightarrow^* v$ iff $t \Downarrow v$.