CS520, Handout 27, 5 December 2006

Congruence Relations

Not in [Pierce]

Let $\mathcal{T}$ be the set of all recursive types, as defined in Handout 26. A binary relation on $\mathcal{T}$ is a subset of $\mathcal{T} \times \mathcal{T}$. If R is a binary relation on $\mathcal{T}$, i.e., $R \subseteq \mathcal{T} \times \mathcal{T}$, it is sometimes convenient to write $T_1 \; R \; T_2$ instead of $(T_1, T_2) \in R$.

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A binary relation $\sim$ on $\mathcal{T}$ is an ***equivalence*** if $\sim$ is *reflexive*, *symmetric* and *transitive*. More formally, $\sim$ is an equivalence if it satisfies the following rules, for all $T, T_1, T_2, T_3 \in \mathcal{T}$:

$$\frac{}{T \sim T} \qquad (\textsc{Refl})$$

$$\frac{T_1 \sim T_2}{T_2 \sim T_1} \qquad (\textsc{Symm})$$

$$\frac{T_1 \sim T_2 \qquad T_2 \sim T_3}{T_1 \sim T_3} \qquad (\textsc{Tran})$$

The equivalence relation $\sim$ partitions $\mathcal{T}$ into ***equivalence classes***. The collection of all equivalence classes is denoted $(\mathcal{T}/\sim)$. Let $\approx$ be another equivalence relation on $\mathcal{T}$. We say $\sim$ ***is smaller than*** $\approx$, or $\approx$ ***is larger than*** $\sim$, if for every $B \in (\mathcal{T}/\sim)$ there is $B' \in (\mathcal{T}/\approx)$ such that $B \subseteq B'$.

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$$\frac{T_1 \sim T_2}{(\mu X.T_1) \sim (\mu X.T_2)} \qquad (\mu\text{-}\textsc{Cong})$$

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Quick Questions:

1. What is the smallest congruence relation on $\mathcal{T}$?
2. What is the largest congruence relation on $\mathcal{T}$?
3. Define two congruences on $\mathcal{T}$ that are incomparable. (Neither one of the two can be the smallest or the largest – why?).

More Questions:

1. Describe the smallest congruence relation $\sim$ on $\mathcal{T}$ satisfying the axiom:

$$\overline{\mu X.T \;\sim\; \mu Y.[X \mapsto Y]T}$$

2. Describe the smallest congruence relation $\sim$ on $\mathcal{T}$ satisfying the axiom:

$$\overline{\mu X.T \;\sim\; [X \mapsto \mu X.T]T}$$

3. Describe the smallest congruence relation $\sim$ on $\mathcal{T}$ satisfying both of the axioms in (1) and (2).