CS520, Handout 29, 7 December 2006

Recursive Types and Subtyping

[Pierce, Section 20.3, beginning of Chapter 21]

Recall from [Pierce, Ch 15-16] and Handouts 23-24-25, the subtyping rules, appropriately adapted:

$$\frac{}{T <: T} \qquad \frac{}{T <: \mathtt{Top}} \qquad \frac{}{\mathtt{Nat} <: \mathtt{Real}}$$

$$\frac{T <: T' \qquad T' <: T''}{T <: T''} \qquad \frac{T_1 <: T_1' \qquad T_2 <: T_2'}{T_1 \times T_2 <: T_1' \times T_2'}$$

$$\frac{T_1 <: T_1' \qquad T_2 <: T_2'}{T_1' \to T_2 <: T_1 \to T_2'}$$

Recall from [Pierce, Ch 15-16] and Handouts 23-24-25, the subtyping rules, appropriately adapted:

$$\frac{}{T <: T} \qquad \frac{}{T <: \mathtt{Top}} \qquad \frac{}{\mathtt{Nat} <: \mathtt{Real}}$$

$$\frac{T <: T' \qquad T' <: T''}{T <: T''} \qquad \frac{T_1 <: T_1' \qquad T_2 <: T_2'}{T_1 \times T_2 <: T_1' \times T_2'}$$

$$\frac{T_1 <: T_1' \qquad T_2 <: T_2'}{T_1' \to T_2 <: T_1 \to T_2'}$$

How can we handle subtyping of recursive types?

We would like to assert that:

$$\mu X.\, \mathtt{Real} \to (\mathtt{Nat} \times X) \quad <: \quad \mu Y.\, \mathtt{Nat} \to (\mathtt{Real} \times Y)$$

Define subtyping contexts as follows: $\Delta \ ::= \ \varnothing \ \mid \ \Delta,\ X <: Y$

The congruence relation $\approx$ was defined in Handouts 27 and 28. The appropriate subtyping rules when recursive types are added:

$$\frac{T \approx T'}{\Delta \vdash T <: T'} \qquad \frac{}{\Delta \vdash T <: \mathtt{Top}} \qquad \frac{}{\Delta \vdash \mathtt{Nat} <: \mathtt{Real}}$$

$$\frac{\Delta \vdash T <: T' \qquad \Delta \vdash T' <: T''}{\Delta \vdash T <: T''} \qquad \frac{\Delta \vdash T_1 <: T_1' \qquad \Delta \vdash T_2 <: T_2'}{\Delta \vdash T_1 \times T_2 <: T_1' \times T_2'}$$

$$\frac{\Delta \vdash T_1 <: T_1' \qquad \Delta \vdash T_2 <: T_2'}{\Delta \vdash T_1' \to T_2 <: T_1 \to T_2'}$$

Define subtyping contexts as follows: $\Delta ::= \varnothing \mid \Delta, X <: Y$

The congruence relation $\approx$ was defined in Handouts 27 and 28. The appropriate subtyping rules when recursive types are added:

$$\frac{T \approx T'}{\Delta \vdash T <: T'} \qquad \frac{}{\Delta \vdash T <: \mathtt{Top}} \qquad \frac{}{\Delta \vdash \mathtt{Nat} <: \mathtt{Real}}$$

$$\frac{\Delta \vdash T <: T' \qquad \Delta \vdash T' <: T''}{\Delta \vdash T <: T''} \qquad \frac{\Delta \vdash T_1 <: T_1' \qquad \Delta \vdash T_2 <: T_2'}{\Delta \vdash T_1 \times T_2 <: T_1' \times T_2'}$$

$$\frac{\Delta \vdash T_1 <: T_1' \qquad \Delta \vdash T_2 <: T_2'}{\Delta \vdash T_1' \to T_2 <: T_1 \to T_2'}$$

So far Δ plays no role. But now we add two new rules:

$$\frac{}{\Delta, X <: Y \vdash X <: Y}$$

$$\frac{\Delta, X <: Y \vdash T <: T'}{\Delta \vdash \mu X.T <: \mu Y.T'}$$ $X \notin FTV(T')$ and $Y \notin FTV(T)$, and X and Y do not occur in Δ

The preceding rules define a proof system to derive judgements of the form $\Delta \vdash T <: T'$. As with any proof system, we need to show its **soundness**. How shall we do this? Essentially, what we want is a way of showing, for each of the rules, if the premises (the judgements above the line) are "true" then the conclusion is also "true" (the judgment below the line). That is, we take soundness of this system to mean that it preserves truth.

We do not have the formal means to assert the truth of a judgement of the form $\Delta \vdash T <: T'$. This is an interesting topic to pursue, but we do not have the time for it. We will argue instead **informally** about "truth" and "soundness", as follows.

If T is a recursive type, let $\lceil T \rceil$ denote the possibly infinite type (or parse tree) obtained by repeatedly unwinding the μ-bindings in T. Let T and T' be closed recursive types, i.e., they do not contain free type variables. We say that $T <: T'$ is **true** if we can somehow assert that $\lceil T \rceil <: \lceil T' \rceil$. We do not further specify in this handout how to formally establish such an assertion, which we leave to be argued informally. **We say the system is sound if it only derives *true* subtyping relationships, $T <: T'$, between closed recursive types T and T'.**

Exercise 1: Show that the system on page 3 of this handout becomes unsound if we change its last rule to:

$$\frac{\Delta, X <: Y \vdash T <: T'}{\Delta \vdash \mu X.T <: \mu Y.T'} \qquad X \notin FTV(T') \text{ and } Y \notin FTV(T)$$

That is, X or Y or both may now occur in Δ. Hint: Suppose $Z <: Y$ occurs in Δ and T contains occurrences of both Z and X.

Exercise 2: Show that the system on page 3 of this handout becomes unsound if we change its last rule to:

$$\frac{\Delta \vdash T <: T'}{\Delta \vdash \mu X.T <: \mu X.T'} \qquad X \text{ does not occur in } \Delta$$

Hint: Consider $T = X \to \mathtt{Nat}$.