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Handout 3 **Assaf J. Kfoury**

Gentzen Systems

We need some preliminary definitions. A *sequent* is a pair of finite (possibly empty) sequences of wff's, say $\Gamma = \gamma_1, \ldots, \gamma_\ell$ and $\Delta = \delta_1, \ldots, \delta_m$, which we write as $\Gamma \mathbin{\vert} \Delta$ or as $\gamma_1, \ldots, \gamma_\ell \mathbin{\vert} \delta_1, \ldots, \delta_m$. The first part Γ is the *antecedent* of the sequent, and the second part Δ the *succedent*. If Γ is empty, we write $\vert\Delta$ for the sequent; if Δ is empty, we write $\Gamma\vert$; and if both are empty, we write $\vert$.

For an intuitive understanding of the axiom scheme and inference rules below, keep in mind the following: If $\ell, m \geqslant 1$, the sequent $\gamma_1, \ldots, \gamma_\ell \mathbin{\vert} \delta_1, \ldots, \delta_m$ will have the same meaning as the wff $\gamma_1 \wedge \cdots \wedge \gamma_\ell \to \delta_1 \vee \ldots \vee \delta_m$. More precisely, given an interpretation $(\mathfrak{A}, s)$, i.e. a structure $\mathfrak{A}$ together with a valuation of the variables $s : V \to |\mathfrak{A}|$, $(\mathfrak{A}, s)$ satisfies the sequent $\gamma_1, \ldots, \gamma_\ell \mathbin{\vert} \delta_1, \ldots, \delta_m$ iff

$$(\mathfrak{A}, s) \models \gamma_1 \wedge \cdots \wedge \gamma_\ell \to \delta_1 \vee \ldots \vee \delta_m$$

This extends to the case when $\ell = 0$ or $m = 0$: If $\ell = 0$, the conjunction $\gamma_1 \wedge \cdots \wedge \gamma_\ell$ is viewed as "true" and the sequent is equivalent to $\delta_1 \vee \ldots \vee \delta_m$; if $m = 0$, the disjunction $\delta_1 \vee \ldots \vee \delta_m$ is viewed as "false" and the sequent is equivalent to $\neg(\gamma_1 \wedge \cdots \wedge \gamma_\ell)$ or $\neg\gamma_1 \vee \cdots \vee \neg\gamma_\ell$; if $\ell = m = 0$, the sequent $\vert$ is unsatisfiable.

About notation

In Kleene's presentation [6], on which this handout is based, as well as in many other presentations of Gentzen systems (e.g. see [4] or [10]), a sequent is written as $\Gamma \boldsymbol{\to} \Delta$ (with a boldface arrow) and our arrow "$\to$" (not boldface) is the same as their "$\supset$". To avoid confusion with our "$\to$", we use "$\vert$" (boldface bar) instead of "$\boldsymbol{\to}$" in sequents. The notation is this handout is similar (but not quite identical) to Dummett's [3], which uses another symbol yet, namely ":", to separate the two parts of a sequent. (We cannot use ":" as the sequent separator because we will need it for another purpose when we consider intuitionistism and its connection to typed λ-calculi.)

In recent years, some people have started to use "$\vdash$" to separate the two parts of a sequent (e.g. see Chapter 5 in [5]). This has been a common practice, for example, when a type-inference system is set up for a typed λ-calculus, where an expression of the form "$A \vdash M : \tau$" can be viewed as a sequent. This is perhaps unfortunate — or, in any case, it goes against the traditional practice of reserving "$\vdash$", a symbol outside the syntax of formal expressions, to assert that a formal expression can be derived using axioms and inference rules. What is a convenient shorthand to say that a "sequent with antecedent Γ and succedent Δ is derivable"? In our notation, we write $\vdash \Gamma \mathbin{\vert} \Delta$, and in Kleene's $\vdash \Gamma \rightarrow \Delta$. But it is clearly confusing to write $\vdash \Gamma \vdash \Delta$, where the two occurrences of $\vdash$ mean two different things.

The symbol "$\mid$" is not in the syntax of wff's, but in the syntax of sequents. The rules of the Gentzen system below transform formal expressions that are sequents into other such formal expressions.

It is easier to see the organization of a Gentzen system by first considering the simpler case of propositional logic. We later point out how to extend the system for the case of first-order logic.

Propositional logic

Γ, Δ, Θ, and Λ range over finite sequences of zero or more wff's. α, β, γ and δ range over the set of wff's. There is only one axiom scheme, and many inference rules. The latter are divided into *logical* and *structural*. All logical rules are "introduction" rules, in the sense that they introduce one of the logical connectives $\rightarrow, \wedge, \vee, \neg$, on the left or on the right of a sequent, and they are thus given the suggestive names $(\rightarrow\mid)$, $(\wedge\mid)$, $(\vee\mid)$, $(\neg\mid)$ (introduction on the left) and $(\mid\rightarrow)$, $(\mid\wedge)$, $(\mid\vee)$, $(\mid\neg)$ (introduction on the right).

- *Axiom scheme* $\alpha \mid \alpha$
- *Logical inference rules*
  - introduction of $\rightarrow$

$$\frac{\Delta \mid \Lambda, \alpha \qquad \beta, \Gamma \mid \Theta}{\alpha \rightarrow \beta, \Delta, \Gamma \mid \Lambda, \Theta} \ (\rightarrow\mid) \qquad\qquad \frac{\alpha, \Gamma \mid \Theta, \beta}{\Gamma \mid \Theta, \alpha \rightarrow \beta} \ (\mid\rightarrow)$$

  - introduction of $\wedge$

$$\frac{\alpha, \Gamma \mid \Theta}{\alpha \wedge \beta, \Gamma \mid \Theta} \ (\wedge\mid) \qquad \frac{\beta, \Gamma \mid \Theta}{\alpha \wedge \beta, \Gamma \mid \Theta} \ (\wedge\mid) \qquad \frac{\Gamma \mid \Theta, \alpha \qquad \Gamma \mid \Theta, \beta}{\Gamma \mid \Theta, \alpha \wedge \beta} \ (\mid\wedge)$$

  - introduction of $\vee$

$$\frac{\alpha, \Gamma \mid \Theta \qquad \beta, \Gamma \mid \Theta}{\alpha \vee \beta, \Gamma \mid \Theta} \ (\vee\mid) \qquad \frac{\Gamma \mid \Theta, \alpha}{\Gamma \mid \Theta, \alpha \vee \beta} \ (\mid\vee) \qquad \frac{\Gamma \mid \Theta, \beta}{\Gamma \mid \Theta, \alpha \vee \beta} \ (\mid\vee)$$

  - introduction of $\neg$

$$\frac{\Gamma \mid \Theta, \alpha}{\neg\alpha, \Gamma \mid \Theta} \ (\neg\mid) \qquad\qquad \frac{\alpha, \Gamma \mid \Theta}{\Gamma \mid \Theta, \neg\alpha} \ (\mid\neg)$$

- *Structural inference rules*
  - weakening

$$\frac{\Gamma \mid \Theta}{\gamma, \Gamma \mid \Theta}\ (\mathtt{W}|) \qquad\qquad \frac{\Gamma \mid \Theta}{\Gamma \mid \Theta, \gamma}\ (|\mathtt{W})$$

  - contraction

$$\frac{\gamma, \gamma, \Gamma \mid \Theta}{\gamma, \Gamma \mid \Theta}\ (\mathtt{C}|) \qquad\qquad \frac{\Gamma \mid \Theta, \gamma, \gamma}{\Gamma \mid \Theta, \gamma}\ (|\mathtt{C})$$

  - exchange

$$\frac{\Delta, \delta, \gamma, \Gamma \mid \Theta}{\Delta, \gamma, \delta, \Gamma \mid \Theta}\ (\mathtt{E}|) \qquad\qquad \frac{\Gamma \mid \Lambda, \gamma, \delta, \Theta}{\Gamma \mid \Lambda, \delta, \gamma, \Theta}\ (|\mathtt{E})$$

  - cut

$$\frac{\Delta \mid \Lambda, \gamma \qquad \gamma, \Gamma \mid \Theta}{\Delta, \Gamma \mid \Lambda, \Theta}\ (\mathtt{Cut})$$

Let G_0 denote the Gentzen system described above for propositional logic. In the examples to follow, a double line in a derivation (with the citation of a rule) stands for zero or more applications of the weakening, contraction, and exchange rules (following the application of the cited rule).

Example 1. For arbitrary wff's α and β, the following is a valid derivation in G_0:

$$\dfrac{\dfrac{\dfrac{\alpha \mid \alpha}{\alpha, \neg\alpha \mid \beta}\ (\neg|)}{\neg\alpha \mid \alpha \to \beta}\ (|\to)}{\mid \neg\alpha \to (\alpha \to \beta)}\ (|\to)$$

Example 2. For an arbitrary wff α, the following are valid derivations in G_0:

$$\dfrac{\dfrac{\alpha \mid \alpha}{\mid \alpha, \neg\alpha}\ (|\neg)}{\mid \alpha \vee \neg\alpha}\ (|\vee) \qquad\qquad \dfrac{\dfrac{\dfrac{\alpha \mid \alpha}{\mid \alpha, \neg\alpha}\ (|\neg)}{\neg\neg\alpha \mid \alpha}\ (\neg|)}{\mid \neg\neg\alpha \to \alpha}\ (|\to)$$

First-order logic

The system for propositional logic is extended by adding rules for the quantifiers and, if an equality symbol $\approx$ is included in the syntax of wff's, axiom schemes and/or rules for equality. The simplest here, for purposes of comparison with the systems in Handouts 1 and 4, is to have only axiom schemes for equality.[1] As in Handout 1 and Enderton's book (page 105), α^x_t is the wff obtained from wff α by substituting every free occurrence of x by the term t, provided the substitution is legal (i.e. no free variable in t is captured by a quantifier in α). Note the proviso "y not free in Γ and Θ" in the $(\mid\forall)$ and $(\exists\mid)$ rules.

- *Inference rules for quantifiers*
  - introduction of $\forall$

$$\frac{\alpha^x_t, \Gamma \mid \Theta}{(\forall x\, \alpha), \Gamma \mid \Theta}\ (\forall\mid) \qquad\qquad \frac{\Gamma \mid \Theta, \alpha^x_y}{\Gamma \mid \Theta, (\forall x\, \alpha)}\ (\mid\forall)$$

$(y$ not free in Γ and $\Theta)$

  - introduction of $\exists$

$$\frac{\alpha^x_y, \Gamma \mid \Theta}{(\exists x\, \alpha), \Gamma \mid \Theta}\ (\exists\mid) \qquad\qquad \frac{\Gamma \mid \Theta, \alpha^x_t}{\Gamma \mid \Theta, (\exists x\, \alpha)}\ (\mid\exists)$$

$(y$ not free in Γ and $\Theta)$

- *Axiom schemes for equality*
  - $\mid\ x \approx x$
  - $x_1 \approx y_1,\ \ldots\ ,\ x_n \approx y_n\ \mid\ f x_1 \cdots x_n \approx f y_1 \cdots y_n$
    (for every function symbol f of arity $n \geq 1$)
  - $x_1 \approx y_1,\ \ldots\ ,\ x_n \approx y_n,\ P x_1 \cdots x_n\ \mid\ P y_1 \cdots y_n$
    (for every predicate symbol P of arity $n \geq 1$)

Let G denote the Gentzen system described above for first-order logic.

Exercise 1. Show that without the proviso "y not free in Γ and Θ" in $(\mid\forall)$ and $(\exists\mid)$ the system is inconsistent, i.e. there is a derivation in G for the unsatisfiable sequent $\mid$.

[1]In [6] the equality symbol in not included in the syntax of wff's. The axiom schemes we choose for equality here are basically those in [4], Section 6.3.

Example 3. For an arbitrary wff α, the following is a valid derivation in G:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha^x_y \mid \alpha^x_y}{\alpha^x_y \mid (\exists x\ \alpha)}\,(\mid\exists) \qquad \dfrac{(\exists x\ \alpha) \mid (\exists x\ \alpha)}{(\exists x\ \alpha), \neg(\exists x\ \alpha) \mid}\,(\neg\mid)}{\alpha^x_y, \neg(\exists x\ \alpha) \mid}\,(\texttt{Cut})}{\dfrac{\neg(\exists x\ \alpha) \mid \neg\alpha^x_y}{\neg(\exists x\ \alpha) \mid (\forall x\ \neg\alpha)}\,(\mid\forall)}\,(\mid\neg)}{\mid \neg(\exists x\ \alpha) \to (\forall x\ \neg\alpha)}\,(\mid\to)$$

In the next theorem, the symbol "$\vdash_H$" is for derivability relative to one of the Hilbert systems in Handout 1 (which all derive precisely the same set of wff's), and "$\vdash_G$" is for derivability relative to G. Γ is an arbitrary finite (possibly empty) sequence of wff's and α an arbitrary wff.

Theorem 1. $\Gamma \vdash_H \alpha$ *if and only if* $\vdash_G \Gamma \mid \alpha$.

Proof: Both directions are straightforward, if somewhat tedious, inductions on derivations. The details are in Section 77 of Kleene's book [6]. Kleene uses a specific Hilbert system, page 82 in [6], which is not the same as any of the systems in Handout 1, but which is readily shown to be equivalent to System E in Handout 1 (and therefore equivalent to all the systems in Handout 1). ■

If Γ in Theorem 1 is a finite sequence of wff's $\gamma_1, \ldots, \gamma_n$ then, by repeated use of the Deduction Theorem (Enderton's, page 111), $\Gamma \vdash_H \alpha$ iff $\vdash_H \gamma_1 \to \cdots \to \gamma_n \to \alpha$. By repeated use of $(\texttt{E}\mid)$ and $(\mid\to)$, we also have that $\vdash_G \Gamma\mid\alpha$ iff $\vdash_G \mid \gamma_1 \to \cdots \to \gamma_n \to \alpha$. Hence, in this case,

$$\vdash_H \gamma_1 \to \cdots \to \gamma_n \to \alpha \qquad \text{iff} \qquad \vdash_G \mid \gamma_1 \to \cdots \to \gamma_n \to \alpha\ ,$$

which makes the symbol "$\mid$" appear redundant, and justifies the practice of some people to discard the symbol "$\vdash_G$" altogether (as asserting derivability) and use "$\vdash_G$" instead of "$\mid$" (as sequent separator) for complete notational symmetry between the Hilbert and Gentzen systems.

It is clear that every derivation $\mathcal{D}$ in G can be organized in the form of a tree, with a sequent inserted at every node of the tree. The *root sequent*, i.e. the one inserted at the root of the derivation tree, is the sequent derived by $\mathcal{D}$. Let us say that two derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ in G are *equivalent* if $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same root sequent. Let G^- denote the system obtained from G by omitting the rule (`Cut`).

Theorem 2 (Gentzen's Hauptsatz[2], or cut-elimination theorem). *Every derivation in G can be effectively transformed into an equivalent derivation in G^- .*

Proof: When there is no equality symbol $\approx$ in the language and no axiom schemes for equality, a proof can be found in [6], Section 78, or in [10], Chapter XII. In the presence of $\approx$, a proof is given in [4], Chapter 6.[3] ■

[2] Nothing mysterious about this name: Hauptsatz means "main theorem" in German.

[3] *Warning*: There are several variants of Gentzen systems in [6], [10] and [4], sometimes with minor differences introduced for reasons of efficiency and/or clarity of exposition. What we here call G is exactly the system $G1$ in [6],

Example 4. The following is a derivation in G^-, which is also equivalent to the derivation in G of Example 3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\alpha^x_y \mid \alpha^x_y}{\alpha^x_y \mid (\exists x\ \alpha)}\,(\mid\exists)}{\alpha^x_y, \neg(\exists x\ \alpha) \mid}\,(\neg\mid)}{\neg(\exists x\ \alpha) \mid \neg\alpha^x_y}\,(\mid\neg)}{\neg(\exists x\ \alpha) \mid (\forall x\ \neg\alpha)}\,(\mid\forall)}{\mid \neg(\exists x\ \alpha) \to (\forall x\ \neg\alpha)}\,(\mid\to)
$$

There are several important applications of Gentzen's Hauptsatz in first-order logic, none discussed in Enderton's book. For example, it can be used to establish Craig's Interpolation Theorem, which in turn is used to establish Beth's Definability Theorem and Robinson's Joint Consistency Theorem (for statements of these theorems and proofs based on cut-elimination, see [4], Chapter 6, or [10], Chapter XV, or [1], Section 9.12). It is worth noting that these three theorems can be established just as elegantly using model-theoretic techniques, without recourse to the syntactic transformation of Gentzen's Hauptsatz (see [2], Section 2.2, or [8], Chapter 22). But more important is the use of Gentzen's Hauptsatz in investigations of consistency results — more on this below.

Restrictions for intuitionism

A Gentzen system for intuitionistic propositional logic (resp. first-order logic) is obtained by omitting just two rules from G_0 (resp. G): $(\mid\neg)$ and $(\mid\mathtt{W})$.

Example 5. The derivation in Example 1 is acceptable intuitionistically, but none of the derivations in Examples 2, 3 and 4, is, because each of the latter uses rule $(\mid\neg)$.

An equivalent restriction of G for intuitionism is given in the next exercise.

Exercise 2. Show that omitting $(\mid\neg)$ and $(\mid\mathtt{W})$ from G is equivalent to requiring that the succedent of every sequent has *at most* one wff, i.e. every sequent is restricted to be of the form $\Gamma\mid$ or the form $\Gamma\mid\beta$. (*Hint*: Consider the easier case of G_0 first. Show by induction that if $\vdash \Gamma\mid\beta_1, \ldots, \beta_m$ without $(\mid\neg)$ and $(\mid\mathtt{W})$, as well as without $(\neg\mid)$, then $m = 1$ — and if without just $(\mid\neg)$ and $(\mid\mathtt{W})$, then $m \leqslant 1$. For the opposite implication, show by induction on derivations that every use of $(\mid\neg)$ or $(\mid\mathtt{W})$ can be eliminated.)

Let G_I denote the system obtained from G by omitting the two rules $(\mid\neg)$ and $(\mid\mathtt{W})$, and let G_I^- denote the system obtained from G_I by omitting in addition the rule $(\mathtt{Cut})$.

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and essentially (but not exactly) the system $\mathcal{G}^*$ in [10], in both cases without any of the parts related to $\approx$. In [4] there is no less than a dozen systems, and among these, LK_e is exactly our G.

Theorem 3 (Gentzen's Hauptsatz for intuitionistic logic). *Every derivation in G_I can be effectively transformed into an equivalent derivation in G_I^- .*

Proof: When there is no equality symbol $\approx$ in the language and no axiom schemes for equality, a proof can be found in [6], Section 78. ■

Gentzen's Hauptsatz and consistency results

For Gentzen, the Hauptsatz was a tool he used to establish the consistency of a particular axiomatization of number theory, commonly called *Peano arithmetic*. Moreover, Gentzen wanted to establish this consistency by proof-theoretic means, without appeal to semantic considerations.

Let P denote Peano arithmetic. The axioms of P are the finitely many axioms of A_M (in Section 3.7 of Enderton's book) in addition to the following *induction axiom scheme*:

$$\varphi(\mathbf{0}) \;\to\; (\; \forall \boldsymbol{v}_1 \; (\varphi(\boldsymbol{v}_1) \;\to\; \varphi(\boldsymbol{S}\boldsymbol{v}_1)) \;) \;\to\; (\; \forall \boldsymbol{v}_1 \; \varphi(\boldsymbol{v}_1) \;)$$

representing infinitely many axioms, one for each wff φ with no free variable other than $\boldsymbol{v}_1$, in the language of the standard model of arithmetic $\mathfrak{N}$. An instance of the induction axiom scheme is an *induction axiom*.[4]

Recall that a set Γ of wff's is consistent if Γ does not derive any contradiction, i.e. if for every wff α it is the case that $\Gamma \not\vdash (\alpha \wedge \neg\alpha)$. Typically, it is not easy to verify such a condition by proof-theoretic means. It is often easier to check the satisfiability of Γ instead which, by soundness and completeness, is equivalent to its consistency.

For example, the finite set A_M is consistent because we recognize that the standard model $\mathfrak{N}$ satisfies every axiom in A_M, thus avoiding the hard work of proving directly that A_M does not derive a contradiction. The same can be said of P, by our understanding of induction over the natural numbers. In fact, based on our common knowledge of the underlying operations and relations of $\mathfrak{N}$, i.e. $<$, $+$, $\cdot$, etc., we are willing to declare the axioms of A_M are "obviously true" in $\mathfrak{N}$. If taken to task, we can of course be more rigorous by applying the methods of Section 2.2 to check that $\mathfrak{N} \models A_M$, and with a little extra work we can also check that $\mathfrak{N} \models P$, but again this uses unformalized set-theoretic reasoning.

This is a common situation: Semantic methods involve a certain amount of set-theoretic as well as intuitive reasoning outside formal proof systems. As a consequence, in relation to questions of consistency, it is sometimes argued that semantic methods are uninformative, or in any case do not provide the right kind of information, or are too precarious to serve as a basis for consistency proofs. It is even argued that, short of a formal proof-theoretic verification, the consistency of a set of wff's should be taken as an explicit unproved assumption.[5]

[4] Peano arithmetic is not discussed in Enderton's book. It is listed in the Index with a reference to page 183, but "Peano arithmetic" is nowhere mentioned on page 183! The induction axiom scheme is mentioned in Exercise 3.1.1 of Enderton's, page 183, but it is restricted to the language of the reduct $\mathfrak{N}_S$, and we do not make such a restriction here.

[5] Shoenfield says that "the main trouble with [a consistency proof for P by means of the standard model] is that

My own bias is to accept the consistency of P (or any other set of axioms) based on semantic, i.e. unformalized set-theoretic and intuitive reasoning. There are criteria of rigor for such reasoning and there is no need to formalize them. But this does not mean that establishing the consistency of P by formal proof-theoretic means is without value, because additional information about P beyond its consistency is also obtained in this way. And perhaps this is the real value of Gentzen's demonstration of the consistency of P.

It is towards this goal that Gentzen developed the sequent calculus and then proved the Hauptsatz. After Gentzen's formal proof of the consistency of P, there were other formal proofs following the same general ideas. Gödel's Second Incompleteness Theorem (in Enderton's book, Section 3.6) can be adapted to say: *If P is consistent, then no wff asserting the consistency of P (by some appropriate encoding) is derivable from P.* By Gödel's result, starting from P as non-logical axioms, the rules of G (or any formal system equivalent to G) are not strong enough to formally prove the consistency of P. Thus, in one way or another, the proof system has to be extended to a more powerful deductive calculus, for example by including inference rules with infinitely many premises or by allowing transfinite induction. However extended, the Hauptsatz remains the key technical result in establishing the consistency of P. Further discussion of these issues can be found in [6], Section 79, and in the Appendix of [7].

it is so uninformative. ... It does not increase our understanding, since nothing goes into it which we did not put into it in the first place" [9], page 214.

Mendelson says that, by using semantic methods, "we have not proved in a rigorous way that the axioms of P are true under the standard interpretation, but have taken it as intuitively obvious" [7], page 107.

Kleene says that "giving a model for the axioms [of P] in intuitive arithmetical terms does not establish beyond all doubt that no contradiction can arise in the theory deduced from the axioms, unless it can also be demonstrated that the reasonings in the theory can be translated into intuitive arithmetical reasonings in terms of the objects used in the model" [6], page 475. The proviso in the excerpt from Kleene applies to A_M, and therefore the consistency of A_M using a restricted form of semantic reasoning, i.e. using what is called "finitary set-theoretic methods", is acceptable to Kleene, but the proviso does not apply to P because showing the satisfiability of any induction axiom in the standard model requires "non-finitary set-theoretic methods". A discussion of "finitary" vs. "non-finitary" methods is also found in Shoenfield's book [9], page 214.

References

[1] Bell, J., and Machover, M., *A Course in Mathematical Logic*, North-Holland, Amsterdam, 1977.

[2] Chang, C.C. and Keisler, H.J., *Model Theory*, North-Holland, Amsterdam, 1973.

[3] Dummett, M., *Elements of Intuitionism*, Oxford University Press, 1977.

[4] Gallier, J.H., *Logic for Computer Science, Foundations of Automatic Theorem Proving*, Harper and Row, New York, 1986.

[5] Girard, J.-Y., Lafont, Y., and Taylor, P., *Proofs and Types*, Cambridge University Press, Cambridge, UK, 1989.

[6] Kleene, S.C., *Introduction to Metamathematics*, Van Nostrand, 1952.

[7] Mendelson, E., *Introduction to Mathematical Logic*, Van Nostrand, 1964.

[8] Monk, J.D., *Mathematical Logic*, Springer-Verlag 1976.

[9] Shoenfield, J.R., *Mathematical Logic*, Addision-Wesley, 1967.

[10] Smullyan, R.M., *First-Order Logic*, Springer-Verlag, New York, 1968.