BU CLA MA 531: Computability and Logic

Fall 1995

Handout 5

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Preliminary Remarks on Intuitionism

A distinctive feature of intuitionism is that it will not accept proofs that are not *constructive*. What is a constructive proof? The best answer is given by examples: Below are three *non*-constructive proofs of well-known results. After the proofs, we point out what makes them non-constructive. These examples are often used when intuitionism is first introduced.

Theorem 1 *There are solutions of $x^y = z$ with x and y irrational numbers and z rational.*

Proof: We know $\sqrt{2}$ is irrational. Moreover, ${\sqrt{2}}^{\sqrt{2}}$ is either rational or irrational. If it is rational, let $x = \sqrt{2}$ and $y = \sqrt{2}$, making $z = x^y$ a rational number. If ${\sqrt{2}}^{\sqrt{2}}$ is irrational, let $x = {\sqrt{2}}^{\sqrt{2}}$ and $y = \sqrt{2}$, so that $z = x^y = ({\sqrt{2}}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2$, which is again a rational number. ■

Theorem 2 (König's Lemma) *Every infinite, finitely branching, tree $\mathcal{T}$ has an infinite path.*

Proof: Using induction, we define an infinite sequence of nodes $x_0, x_1, \ldots$, forming an infinite path in $\mathcal{T}$. At stage 0 of the induction, let x_0 be the root of $\mathcal{T}$, which has infinitely many successors by the hypothesis that $\mathcal{T}$ is infinite. At stage $n \geq 1$, assume we have already selected nodes $x_0, x_1, \ldots, x_{n-1}$ so far, forming a path of length $n-1$, such that x_{n-1} has infinitely many successors. By hypothesis, $\mathcal{T}$ is finitely branching, which implies x_{n-1} has only finitely many immediate successors. Hence, one of the immediate successors of x_{n-1}, say y, must have infinitely many successors. Define x_n to be y, which has infinitely many successors in $\mathcal{T}$, and proceed to stage $n+1$ of the induction. ■

Another way of stating König's Lemma (KL) is to say: *If a finitely branching tree has arbitrarily long finite paths, then it has an infinite path.* Stated this way, it evokes some connection with the compactness theorem in classical logic: If a set Γ of wff's is finitely satisfiable, then Γ is satisfiable. This is indeed the case, as it can be shown that KL and the compactness theorem basically assert the same thing.[1]

[1] To see the connection in the case of propositional logic, consider the collection of all truth assignments to the propositional variables $A_1, A_2, \ldots, A_n, \ldots$, organized as a single infinite full binary tree, call it $\mathcal{T}$, i.e. at the n-th level of $\mathcal{T}$ a left (resp. right) branch corresponds to assigning T (resp. F) to variable A_n. Given a set Σ of propositional wff's, we define another binary tree $\mathcal{T}^{\Sigma}$ from $\mathcal{T}$: Given infinite path $\pi = t_1 t_2 \cdots t_n \cdots$ in $\mathcal{T}$, where each t_n is T or F, let k be the smallest integer (if any) such that the truth assignment corresponding to π does *not* satisfy some wff in Σ; if such a k exists, delete from $\mathcal{T}$ all paths extending the finite path $t_1 t_2 \cdots t_k$. The resulting $\mathcal{T}^{\Sigma}$ contains some finite paths (possibly none) and some infinite paths (possibly none). Now, Σ is finitely satisfiable iff $\mathcal{T}^{\Sigma}$ has arbitrarily long finite paths (equivalently, which is easier to see, there is an unsatisfiable finite subset of Σ iff there is a finite bound on the length of all paths in $\mathcal{T}^{\Sigma}$) and Σ is satisfiable iff $\mathcal{T}^{\Sigma}$ has an infinite path.

Another connection with the compactness theorem in logic is the Bolzano-Weierstrass (BW) theorem in real analysis. There are different formulations of this theorem. We give one which makes plain the non-constructive nature of the proof. (Another formulation is given in Exercise 5, page 173, in Enderton's book.)

Theorem 3 (Bolzano-Weierstrass) *Every infinite subset S of the closed interval $[a, b]$ of real numbers contains a convergent infinite sequence.*[2]

Proof: We construct an infinite nested chain of intervals $[a_n, b_n]$, each containing infinitely many elements of S, by induction on $n \geq 0$. First, let $a_0 = a$ and $b_0 = b$. Proceeding inductively, for arbitrary $n + 1 \geq 1$:

(1) If $[a_n, (a_n + b_n)/2]$ contains infinitely many elements, let $a_{n+1} = a_n$ and $b_{n+1} = (a_n + b_n)/2$.

(2) If $[a_n, (a_n + b_n)/2]$ contains finitely many elements, let $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = b_n$.

This is a strict nested chain of intervals with a non-empty intersection. (It is "strict" in the sense that no two consecutive intervals are equal.) Let x be an element in this intersection. The sequence $a_0, a_1, \ldots, a_n, \ldots$ or the sequence $b_0, b_1, \ldots, b_n, \ldots$ must contain infinitely many distinct elements, and both converge to x. ■

The three preceding proofs show the existence of something without providing the means to find it. In the first proof, one of two specific solutions of the equation is shown to be true, but no effective method is given to determine which.

In the proof of KL, we prove by induction that a disjunction is true, but because we do not determine which immediate successor node has infinitely many nodes below it, we do not actually have a construction for (i.e. an algorithm to generate) the infinite path we prove to exist.

The proof of BW seems to specify a construction, but because it does not provide a way of deciding whether case (1) or case (2) holds, such a construction cannot in fact be carried out.

In all three cases, what pushes the argument through to its conclusion is an appeal to the *law of excluded middle*, which says that for every assertion A, either A is true or $\neg A$ is true, even though there may be no effective way of deciding which. This is why we say that these proofs are not *constructive*.

From an intuitionistic point of view, this invalidates the three preceding proofs as well as many other proofs in classical mathematics. It also invalidates many proofs in classical logic, such as the proofs of compactness,[3] completeness, and many other results at the foundation of classical logic. There are issues of compactness and completeness in intuitionism, to be sure, but these have to be understood differently and established differently.

[2] If you had a course in analysis, you will recall this is equivalent to the property that the real numbers form a *compact space*. And, indeed, it can be shown this is the same phenomenon encountered in the compactness theorem in logic.

[3] This should be clear from the connection indicated earlier between the compactness theorem and KL and BW.

Note that the notion of a constructive proof is not restricted to intuitionism and makes perfect sense in the context of classical mathematics too. The distinction between constructive and non-constructive proofs naturally arises in classical mathematics whenever we want to prove an existential statement or a disjunctive statement. A constructive proof of the same theorem, or what may be perceived as the same theorem (more on this below), is generally more informative than a non-constructive one: Not only the existence of something is established, but an effective method (an algorithm) is provided to determine it.

The preceding suggests that intuitionistic logic, as a system of reasoning, ought to be favored over classical logic. Indeed, if constructive proofs are more informative, why should we ever content ourselves with non-constructive proofs? The answer would be "never" if intuitionism were a clear winner in all respects — but it isn't, as something is often lost by abandoning some of the tenets of classical logic (such as the law of excluded middle and others). For one thing, intuitionism is not a single system of reasoning: There are variations within intuitionism, each advocating a different way of relaxing restrictions imposed by the rejection of the law of excluded middle. These should not preoccupy us here, as they become quite technical and result in different approaches to constructive mathematics (e.g. see [3]).

Moreover, a constructive proof of the same fact can be considerably more complicated. For example, concerning the first theorem above, there is a constructive proof for it, but which requires a deeper study of the numbers $\sqrt{2}$ and $\sqrt{2}^{\sqrt{2}}$. (In fact it can be shown that $\sqrt{2}^{\sqrt{2}}$ is irrational.[4])

What about constructive proofs for KL and BW? This points to another difficulty: A rejection of non-constructive proofs comes together with a different interpretation of formal statements. Both KL and BW mention an "infinite" object in their statements. In classical logic, we assume the existence (or pre-existence) of infinite sets as finished (or completed) entities. Intuitionism rejects this view and qualifies a set X as "infinite" only if there is a way of effectively generating the members of X without ever having to stop, and it allows operations (themselves required to be constructive) on X only if they can be carried out without ever having to list (or presume the existence of) all the members of X. The set $\mathbb{N}$ of natural numbers can be viewed infinite in this sense, and classical logic can go along with intuitionism in this case.[5] But the infinity of the set $\mathbb{R}$ of reals has to mean two different things for classical and intuitionistic logic. What is "infinite" intuitionistically is "infinite" classically, but not necessarily the other way around.

Hence, it is not only a matter of choosing between a constructive proof and a non-constructive proof of the same theorem, but also of interpreting formal statements differently. Although the intuitionistic interpretation of a formal statement may convey more information than the classical one, it is this extra information packed into the same concept that often entails a more complicated and less transparent definition of that concept. For example, in relation to the concept of "equality

[4]More generally, it can be shown constructively that if $a \notin \{0, 1\}$, a algebraic, and b irrational algebraic, then a^b is irrational. See page 8 in [3] and appropriate references therein.

[5]The effective enumerability of $\mathbb{N}$ is the starting point of recursion theory and all logicians, whether classical or intuitionistic, are comfortable with it.

between real numbers", Kleene points out the following (page 53 in [2]):

> In the intuitionistic theory of the continuum, we cannot affirm that any two real numbers a and b are either equal or unequal. Our knowledge about the equality or inequality of a and b can be more or less specific. By $a \neq b$, it is meant that $a = b$ leads to a contradiction, while $a \# b$ is a stronger kind of inequality which means that one can give an example of a rational number which separates a and b. Of course $a \# b$ implies $a \neq b$. But there are pairs of real numbers a and b for which it is not known that either $a = b$ or $a \neq b$ or $a \# b$. It is clear that such complications replace the classical theory of the continuum by something less perspicuous in form.

Back to the question of whether there are constructive proofs for KL and BW: Understood constructively, both KL and BW fail. For the first counterexample below, note there are $2^{\aleph_0}$ binary trees (why?) but only $\aleph_0$ of them can be effectively generated (why?).

Constructive counterexample for KL: *There is an effectively generated binary tree which contains infinite paths but none of its infinite paths can be effectively generated.*[6]

The next counterexample mentions "computable" real numbers. This is a restriction on the classical definition of real numbers that makes the notion acceptable intuitionistically. Basically, a real number r is said to be *computable* (or also *recursive*) if there is an effective procedure to generate the numerals (read from left to right) in the decimal expansion of r.

Constructive counterexample for BW: *There is an effectively generated, strictly increasing sequence of rationals in the interval* $[0, 1]$ *which does not converge to any computable real number.*[7]

Exercise 1. For simplicity, restrict attention to binary trees. The contrapositive of KL is sometimes called the Fan Theorem (FT) which asserts: *Every well-founded binary tree is finite.*[8] Classically, KL and FT are equivalent, but not intuitionistically. In fact, FT is accepted intuitionistically, even though KL is not. What is the explanation for this apparent inconsistency? (Not every form of taking the contrapositive is rejected by intuitionism, so you have to be careful in your answer.) ■

[6] This is a paraphrase of a more precise result in recursion theory: *There is a primitive recursive tree R such that (1) for every total recursive function $f : \mathbb{N} \to \{0, 1\}$ there is $n \in \mathbb{N}$ such that $\bar{f}(n) \notin R$, and yet also (2) for every $n \in \mathbb{N}$ there is a path t of length n such that $t \in R$.* Take a "path" to be a binary string, a "binary tree" to be a prefix-closed set of binary strings, and $\bar{f}(n)$ to denote the string $f(0)f(1)\cdots f(n)$. A proof of this result can be found in [1], Ch. IV, Section 5. A discussion of the same is also in [3], Ch. 4, Section 7.

[7] There is a more general result asserting the existence of the so-called "Specker sequences", which implies the result here. The proof along with appropriate definitions can be found in [1], Ch. IV, Section 4, or in [3], Ch. 5, Section 4.

[8] A *well-founded* tree is one without infinite paths.

References

[1] Beeson, M.J., *Foundations of Constructive Mathematics*, Springer-Verlag, 1980.

[2] Kleene, S.C., *Introduction to Metamathematics*, Van Nostrand, 1952.

[3] Troelstra, A.S. and van Dalen, D., *Constructivism in Mathematics*, North-Holland, Amsterdam, 1988.