BU CLA MA 531 Computability and Logic

Fall 1995

Solutions for Problem Set 1

Problem 2.1.2.

Neither a nor b is a member of every set.

$$\neg\forall s\ a{\in}s \wedge \neg\forall b\ {\in}s$$

or, formally,

$$\neg(\neg\forall s(\in as) \to \forall s(\in bs))$$

Problem 2.1.6

Nobody likes everybody.

$$\forall x\exists y\neg Lxy$$

Problem 2.1.8

Free variables appear in **bold**.

$$((\exists v_1(Pv_1)) \wedge (P\boldsymbol{v}_2))$$

$$(((\forall v_1(Av_1)) \wedge (B\boldsymbol{v}_2)) \to ((\exists v_3(\neg Cv_3)) \vee (D\boldsymbol{v}_4)))$$

Problem 2.2.2.

Define a structure $\mathcal{A}$ as follows:

$|\mathcal{A}|$ = the set of natural numbers.

$P^{\mathcal{A}}$ = the set of ordered pairs (m, n) such that $m \leq n$.

Under $\mathcal{A}$, (a) and (b) are true, since $\leq$ is transitive and antisymmetric on the natural numbers. However, (c) is false since the antecedent is true but the conclusion is false. Thus (c) is not logically implied by (a) and (b).

Now define $\mathcal{A}$ as follows.

$|\mathcal{A}|$ = the set of integers.

$P^{\mathcal{A}}$ = the set of ordered pairs (m, n) such that $m = qn$ for some integer q.

Under $\mathcal{A}$, (a) is true. (c) asserts that if each integer is a multiple of some integer, then there exists an integer that is a factor of every other integer: this is trivially true, since the conditional is true (1 is a factor of every integer).

(b), on the other hand, is false whenever m is the additive inverse of n (in which case $q = -1$).

Thus (b) is not logically implied by (a) and (c).

Finally, define $\mathcal{A}$ as follows:

$|\mathcal{A}|$ = the set of nodes of a finite, directed, acyclic graph G.

$P^{\mathcal{A}}$ = the set of ordered pairs (m, n) such that (m, n) is an edge of G.

Under $\mathcal{A}$, (a) is clearly false. (b) is trivially true since (x, y) and (y, x) cannot both be edges of an acyclic graph. (c) is also true, since the antecedent is impossible for G.

Thus (a) is not logically implied by (b) and (c).

We conclude that none of (a), (b), and (c) is logically implied by the other two.

Problem 2.2.12.

$[\,0, \infty\,)$ is definable in $\mathcal{R}$ by $\exists v_1\ v_1 \cdot v_1 = v_2$.

$\{\,2\,\}$ is definable in $\mathcal{R}$ by $(v_1 \cdot v_1 \approx v_1 + v_1) \wedge \neg(v_1 \approx v_1 + v_1)$.

Problem 2.2.18a.

Let $\mathcal{A}$ be a substructure of $\mathcal{B}$, let φ be a universal formula, and let ψ be an existential formula. Then we have,

$$
\begin{aligned}
(\mathcal{B}, s) \models \varphi \quad &\Rightarrow \quad (\mathcal{B}, s) \models \forall x_1 ... \forall x_n \theta \text{ for some quantifier-free } \theta \\
&\Rightarrow \quad (\mathcal{B}, s(x_1|d_1, ..., x_n|d_n)) \models \theta \text{ for every } d_1, ..., d_n \in |\mathcal{B}| \\
&\Rightarrow \quad (\mathcal{A}, s(x_1|d_1, ..., x_n|d_n)) \models \theta \text{ for every } d_1, ..., d_n \in |\mathcal{A}|, \quad \text{since} \quad |\mathcal{A}| \subseteq |\mathcal{B}| \\
&\Rightarrow \quad (\mathcal{B}, s) \models \forall x_1 ... \forall x_n \theta \\
&\Rightarrow \quad (\mathcal{B}, s) \models \varphi
\end{aligned}
$$

and we also have,

$$
\begin{aligned}
(\mathcal{A}, s) \models \psi \quad &\Rightarrow \quad (\mathcal{A}, s) \models \exists x_1 ... \exists x_n \theta \text{ for some quantifier-free } \theta \\
&\Rightarrow \quad (\mathcal{A}, s(x_1|d_1, ..., x_n|d_n)) \models \theta \text{ for some } d_1, ..., d_n \in |\mathcal{A}| \\
&\Rightarrow \quad (\mathcal{B}, s(x_1|d_1, ..., x_n|d_n)) \models \theta \text{ for some } d_1, ..., d_n \in |\mathcal{B}| \\
&\Rightarrow \quad (\mathcal{B}, s) \models \exists x_1 ... \exists x_n \theta \\
&\Rightarrow \quad (\mathcal{B}, s) \models \psi
\end{aligned}
$$

Problem 2.2.18b

$\forall x Px$ is not logically equivalent to any existential sentence ψ.

Proof: Let $\mathcal{A}$ be an arbitrary structure such that $\mathcal{A} \models \forall x Px$, and let $\mathcal{B}$ be an extension of $\mathcal{A}$ such that there exists some z$\in$ $|\mathcal{B}|$-$|\mathcal{A}|$ and $z \notin P^{\mathcal{B}}$. Then B$\not\models \forall x Px$.

But because ψ is existential and $\mathcal{A}$ is a substructure of $\mathcal{B}$, we have $\mathcal{B} \models \psi$ by the Homomorphism Theorem.

Since $\mathcal{B} \not\models \forall x Px$ and $\mathcal{B} \models \psi$, $\forall x Px$ is not logically equivalent to ψ.

A symmetrical argument proves that $\exists x Px$ is not logically equivalent to any universal sentence.

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