BU CLA MA 531 Computability and Logic

Fall 1995

Solutions for Problem Sets 10 and 11

Problem 3.5.4

Let P be the set of all prime numbers and c a fresh constant. There are $2^{|\aleph_0|}$ subsets of P. For every $Q \subseteq P$, define

$$T_Q = Th\ \Re \bigcup \{\ \exists x(c \approx q \cdot x) : q \in Q\ \} \bigcup \{\ \forall x(c \not\approx r \cdot x) : r \in P - Q\ \}.$$

Every finite subset of T_Q is satisfiable, so T_Q is satisfiable by Compactness. T_Q is also consistent, by Soundness. Thus there exists a model of T_Q.

Take any two models $(\mathcal{A}, c^{\mathcal{A}}) \models T_Q$ and $(\mathcal{B}, c^{\mathcal{B}}) \models T_R$, with $Q \neq R$. $(\mathcal{A}, c^{\mathcal{A}}) \not\cong (\mathcal{B}, c^{\mathcal{B}})$, so $\mathcal{A} \not\cong \mathcal{B}$.

Problem 3.8.1

If we were compelled to argue intuitionistically and we did not know of the recent proof of Fermat's last theorem, then we would claim that f is not computable (recursive) in the absence of any procedure to compute it. Once we learn of such a procedure, we retract our claim and declare the function recursive. By classical logic f is a constant function whether we know its value or not, and so is recursive.

Similarly, $g(n)$ can be determined by inspection for $n = 0$ and $n = 1$ but no one has devised a procedure for computing g for arbitrary n. Classically, g is recursive, for consider the statement–

$$(\exists k)[\text{every run of consecutive 7's has length } \leq k]$$

This is either true or false (arguing classically). If false, then $g(n) = 0$ for all n. If true, take the last k witnessing the truth of the statement, then $g(n) = 0$ if $n \leq k$; otherwise $g(n) = 1$.

The intuitionist will claim that g is not recursive (until he discovers a procedure to compute it, at which point he must declare that his original claim was wrong, and he will envy the classical logician who never has to retract a claim of recursiveness).

Problem 3.8.2a

Let $f : \mathcal{N}^m \rightarrow \mathcal{N}$ be a recursive partial function. When viewed as a relation, f is recursively enumerable. Let Q be this relation; that is, $< \vec{a}, b > \in Q$ iff $f(\vec{a}) = b$.

Generate the members of Q, and for each member output $(< a_0, \dots a_m, b >)_{m+1}$. The output is exactly the range of f.

Problem 3.8.2b

Let $f : \mathcal{N}^m \rightarrow \mathcal{N}$ be a nondecreasing total recursive function. If the range of f is finite, then trivially it is recursive, so assume it is infinite. We will show that the range is still recursive by giving an effective procedure for deciding whether an arbitrary $k \in \mathcal{N}$ is in the range.

We effectively generate (encodings of) m-tuples and compute f on each. (By Church's Thesis, f is computable.) Eventually, since the range of f is infinite, we must compute a value that is greater than or equal to k. If the value is greater than k, then k is not in the range of f, as f is nondecreasing; if the value equals k, then k *is* in the range of f.

Problem 3.8.3

If $A = \{a_0, \ldots a_{n-1}\}$ is finite, define a total recursive function $f(k) = a_{k \ mod \ n}$. The range of f is exactly A.

For infinite A, consider the recursive relation Q whose domain is A. We induce an ordering on Q as follows: Let Q_1 be the first pair of $\mathcal{N} \times \mathcal{N}$ that is also in Q (assuming here the ordering on $\mathcal{N} \times \mathcal{N}$ that is obtained by the usual dovetailing procedure: (0,0), (0,1), (1,0), (0,2), (1,1), (2,0), etc.), let Q_2 be the next pair of $\mathcal{N} \times \mathcal{N}$ that is also in Q, and so on.

Define a total recursive function $f(k) = a$, where a is the first element of Q_k. The range of f is exactly the domain of Q, which is A.

To obtain an infinite recursive subset A' of A, construct $Q \subseteq Q'$ as follows:

(1) $Q_0 = < a_0, b_0 >$ is in Q'.

(2) for $k > 0$, $Q_k = < a_k, b_k >$ is in Q' iff $a_k > a_{k-1}$.

Informally, we have removed from Q any pair whose first element is not greater than the first element of every preceding pair. For example, if Q begins with (0,1), (1,2), (2,1), (1,4), (5,0), (1,6), (7,0), then Q' begins with (0,1), (1,2), (2,1), (5,0), (7,0). Note that Q' is infinite, since the domain of Q contains an infinite number of *distinct* elements (dom $Q = A$).

The domain of Q' is A', and $A' \subseteq A$. To show that A' is recursive, we give an effective procedure for deciding if an arbitrary $k \in \mathcal{N}$ is in A': enumerate the members of Q' until either k or a number larger than k appears as the first element of a pair, and decide Yes/No respectively. The procedure requires at most $k + 1$ comparisons.

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*(**3.5.4** by A.J. Kfoury)*
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