BU CLA MA 531 Computability and Logic

Fall 1995

Solutions for Problem Set 12

Prove Gilvenko's Theorem

Suppose $\neg\neg\Gamma \vdash^i \neg\neg\varphi$, and observe that $\vdash^c \neg\neg\varphi \to \varphi$, for by $[\neg\varphi]^1$ and $[\neg\neg\varphi]^2$ with $\neg E$ we have $\bot$, then with RAA_1 (or $\to I_1$) we have φ, and finally with $\to I_2$ we have $\neg\neg\varphi \to \varphi$.

Now take $\neg\neg\varphi$ and $\neg\neg\varphi \to \varphi$ with $\to E$ to obtain φ. We have shown $\neg\neg\Sigma \vdash^c \varphi$, where $\Sigma = \{\psi_1, , \ldots, \psi_k\}$ is the set of uncancelled hypotheses in the derivation of $\neg\neg\varphi$.

$$\begin{aligned}
\{\neg\neg\psi_1, \ldots, \neg\neg\psi_k\} \vdash^c \varphi \quad &\Rightarrow \quad \{\neg\neg\psi_2, \ldots, \neg\neg\psi_k\} \vdash^c \ \neg\neg\psi_1 \to \varphi \text{ (Deduction Theorem)} \\
&\Rightarrow \quad \{\neg\neg\psi_2, \ldots, \neg\neg\psi_k\} \vdash^c \ \psi_1 \to \varphi \text{ (Tautological Equivalence)} \\
&\Rightarrow \quad \{\psi_1, \neg\neg\psi_2, \ldots, \neg\neg\psi_k\} \vdash^c \varphi \text{ (Modus Ponens)} \\
&\ \ \vdots \\
&\Rightarrow \quad \{\psi_1, \ldots, \psi_k\} \vdash^c \varphi \\
&\Rightarrow \quad \Sigma \vdash^c \varphi \\
&\Rightarrow \quad \Gamma \vdash^c \varphi \text{ (since } \Sigma \subseteq \Gamma.)
\end{aligned}$$

Conversely, suppose $\Gamma \vdash^c \varphi$. We will take $\to$ and $\bot$ as our only connectives, as these comprise a complete set. The symbol $\vdash_n$ will denote a derivation of length n. We procede by induction on the length of derivations.

Basis: $\Gamma \vdash_0 \varphi$. Then $\varphi \in \Gamma$ and $\neg\neg\varphi \in \neg\neg\Gamma$. Thus $\neg\neg\Gamma \vdash^i \neg\neg\varphi$.

Case 1: The last rule used is $\to I$.

Let $\varphi = (\alpha \to \beta)$. Suppose $[\alpha]^j$ is an uncancelled hypothesis in the derivation $\Gamma \vdash_n \beta$. With $\to I_j$ we have $\Gamma \vdash_{n+1} (\alpha \to \beta)$. By the inductive hypothesis, $\{\neg\neg\Gamma, \neg\neg\alpha\} \vdash^i \neg\neg\beta$, so with $\to I$ we have $\neg\neg\Gamma \vdash^i (\neg\neg\alpha \to \neg\neg\beta)$. Appealing to Lemma 1.4 we obtain $\neg\neg\Gamma \vdash^i \neg\neg(\alpha \to \beta)$.

Case 2: The last rule used is $\to E$.

Let $[\tau]^j$ be an uncancelled hypothesis in the derivation $\Gamma \vdash_n \tau \to \varphi$. With $\to E_j$ we have $\Gamma \vdash_{n+1} (\alpha \to \beta)$. Exploiting the inductive hypothesis and Lemma 1.4 precisely as above, we obtain the desired result.

Case 3: The last rule used is $\bot$.

φ is an arbitrarily chosen formula. Suppose $\Gamma \vdash_n \bot$. Then with $\bot$-rule we have $\Gamma \vdash_{n+1} \varphi$. By the inductive hypothesis,

$$\begin{aligned}
\neg\neg\Gamma \vdash^i \neg\neg\bot \quad &\Rightarrow \quad \neg\neg\Gamma \vdash^i \neg(\bot \to \bot) \\
&\Rightarrow \quad \neg\neg\Gamma \vdash^i (\bot \to \bot) \to \bot
\end{aligned}$$

$$\Rightarrow \quad \neg\neg\Gamma \vdash^i \perp$$
$$\Rightarrow \quad \neg\neg\Gamma \vdash^i \varphi$$

Case 4: The last rule used is RAA.

φ is again arbitrary, and $[\varphi]^j$ is an uncancelled hypothesis in the derivation $\Gamma \vdash_n \perp$. Then with RAA_j we have $\Gamma \vdash_{n+1} \varphi$. Reasoning as in case 3, we obtain the desired result.

Show that the system of natural deduction is consistent

Let $\mathcal{D}$ be a derivation of φ with no uncancelled hypotheses. Assume without loss of generality that $\mathcal{D}$ is in normal form, and consider any track of $\mathcal{D}$ containing φ (at least one such track must exist).

Suppose $\varphi = \perp$. Then φ must be the conclusion of an application of $\neg E$. Conclude by Lemma C that the track consists only of elimination rules.

How is the hypothesis of this track to be cancelled? The only elimination rules that can be used to cancel hypotheses are $\vee E$ and $\exists E$, but these rules only cancel hypotheses that occur in the subderivation of a minor premise, of which there are none in this track (for otherwise the track would terminate at that point and not include φ).

Conclude $\varphi \neq \perp$. Natural deduction is consistent.

Drue Coles
Created: 95.12.04