BU CLA MA 531 Computability and Logic

Fall 1995

Solutions for Problem Set 2

Problem 2.2.15

We define a function h on the set of prime numbers as follows.

$$h(p) = \begin{cases} 3 & \text{if } p = 2 \\ 2 & \text{if } p = 3 \\ p & \text{otherwise} \end{cases}$$

We then extend h to every $n \in \mathcal{N}$ as follows:

If the prime decomposition of n is $p_1^{1_1}, p_2^{i_2}, \ldots, p_j^{i_j}$ for some primes $p_1, p_2, \ldots, p_j \geq 2$ and exponents $i_1, i_2, \ldots, i_j \geq 1$, then let $h(n) = (h(p_1))^{i_1} \ldots (h(p_j))^{i_j}$. To complete the definition of h, let $h(0) = 0$ and $h(1) = 1$ (0 and 1 are not primes, nor do they have prime decompositions in the form given above).

Multiplication on $\mathcal{N}$ is clearly preserved by h, thus h is an automorphism of $(\mathcal{N}, \cdot)$.

Now suppose that addition is definable in $(\mathcal{N}, \cdot)$ by some relation R. Then we have–

$$(3,4,7) \in \mathrm{R}$$

But $(h(3), h(4), h(7)) = (2, 9, 7) \notin \mathrm{R}$, so we must abandon the supposition that R exists.

Problem 2.2.21

Let y be a variable that does not occur in α, and let β be a formula obtained from α by replacing every free occurrence of x with y. Then–

$$\exists! x\alpha \qquad \text{is logically equivalent to} \qquad \exists x(\alpha \land \forall y(\beta \rightarrow x \approx y))$$

Problem 2.4.8

It suffices by rule T to show

$$\vdash (\alpha \rightarrow \exists x\beta) \rightarrow \exists x(\alpha \rightarrow \beta)$$

and

$$\vdash (\exists x(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \exists x\beta)$$

We prove the former:

$$
\begin{array}{rll}
\alpha \ \vdash \ \neg\beta \rightarrow \neg(\alpha \rightarrow \beta) & \text{Theorem 24B} & (1) \\
\alpha \ \vdash \ \forall x(\neg\beta \rightarrow \neg(\alpha \rightarrow \beta)) & \text{1; Gen} & (2) \\
\forall x(\neg\beta \rightarrow \neg(\alpha \rightarrow \beta)) \ \rightarrow \ (\forall x\neg\beta \rightarrow \forall x\neg(\alpha \rightarrow \beta)) & \text{Ax 3} & (3) \\
\alpha \vdash \forall x\neg\beta \ \rightarrow \ \forall x\neg(\alpha \rightarrow \beta) & \text{2, 3; MP} & (4) \\
\{\alpha,\ \forall x\neg\beta\} \ \vdash \ \forall x\neg(\alpha \rightarrow \beta) & \text{4; Ded} & (5) \\
\{\alpha, \neg\forall x\neg(\alpha \rightarrow \beta)\} \ \vdash \ \neg\forall x\neg\beta & \text{5; Corollary 24D} & (6) \\
\{\alpha, \exists x(\alpha \rightarrow \beta) \ \vdash \ \exists x\beta & \text{Same as preceding} & (7) \\
\exists x(\alpha \rightarrow \beta) \ \vdash \ \alpha \rightarrow \exists x\beta & \text{7; Ded} & (8) \\
\vdash \exists x(\alpha \rightarrow \beta) \ \rightarrow \ (\alpha \rightarrow \exists x\beta) & \text{8; Ded} & (9)
\end{array}
$$

We establish the corollary by replacing α, β in the (9) with their negations.

$$
\begin{array}{rcl}
\vdash (\neg\alpha \rightarrow \exists x\neg\beta) & \leftrightarrow & \exists x(\neg\alpha \rightarrow \neg\beta) \\
\vdash (\neg\alpha \rightarrow \neg\forall x\neg(\neg\beta)) & \leftrightarrow & \exists x(\neg\alpha \rightarrow \neg\beta) \\
\vdash (\neg\alpha \rightarrow \neg\forall x\beta) & \leftrightarrow & \exists x(\neg\alpha \rightarrow \neg\beta) \\
\vdash (\forall x\beta \rightarrow \neg(\neg\alpha)) & \leftrightarrow & \exists x(\beta \rightarrow \alpha) \\
\vdash (\forall x\beta \rightarrow \alpha) & \leftrightarrow & \exists x(\beta \rightarrow \alpha)
\end{array}
$$

Problem 2.4.10

$$\forall x\forall y Pxy \vdash \forall y\forall x Pyx$$

$$
\begin{array}{rl}
\vdash \ (\forall x\forall y Pxy) \rightarrow \ Pc_1c_2 & \text{Ax 2 twice} \\
\forall x\forall y Pxy \ \vdash \ Pc_1c_2 & \text{Deduction Theorem} \\
\forall x\forall y Pxy \ \vdash \ \forall x Pc_1x & \text{Corollary 24G} \\
\forall x\forall y Pxy \ \vdash \ \forall y\forall x Pyx & \text{Corollary 24G}
\end{array}
$$

Problem 2.4.11

$$\vdash \forall x \forall y \forall z (x \approx y \to y \approx z \to x \approx z)$$

$\vdash y \approx x \to (y \approx z \to x \approx z)$	Ax 6	(1)
$\vdash x \approx y \to (x \approx x \to y \approx x)$	Ax 6	(2)
$\vdash x \approx x$	Ax 5	(3)
$\vdash x \approx y \to y \approx x$	2, 3; rule T	(4)
$\vdash x \approx y \to (y \approx z \to x \approx z))$	1, 4; rule T	(5)
$\vdash \forall x \forall y \forall z (x \approx y \to (y \approx z \to x \approx z))$	5; gen^2	(6)

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