BU CLA MA 531 Computability and Logic

Fall 1995

Solutions for Problem Set 3

Problem 2.5.4a

Let $\mathcal{A}$ be a finite structure and φ an atomic formula with n free variables. In order to decide whether $(\mathcal{A}, s) \models \varphi$, we construct a truth table for each of the $n^{|\mathcal{A}|}$ possible valuations. If φ is satisfied by each valuation, then $(\mathcal{A}, s) \models \varphi$. Otherwise, $(\mathcal{A}, s) \not\models \varphi$.

We may then show by induction on an arbitrary (possibly nonatomic and quantified) wff σ that it can be decided whether or not $\mathcal{A} \models \sigma$.

Problem 2.5.4b

The enumerability of the set of finite structures is established below in (**2.5.5a**).

Let $\sigma_1, \sigma_2, \ldots$ be an enumeration of sentences, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be an enumeration of finite structures.

First we test $\mathcal{A}_j \models \sigma_k$ for all $j + k = 2$ (there is only one such possibility: $\mathcal{A}_1 \models \sigma_1$). We then test $\mathcal{A}_j \models \sigma_k$ for all $j + k = 3$ (that is, $\mathcal{A}_1 \models \sigma_2$ and $\mathcal{A}_2 \models \sigma_1$). We continue testing $\mathcal{A}_j \models \sigma_k$ for $j + k = 4, 5, \ldots$ Whenever the result is positive, we output σ_k since it is modeled by the finite structure $\mathcal{A}_j$. Each sentence modeled by a finite structure will eventually appear on the output list.

Problem 2.5.5a

Let $\mathcal{F}_k$ be the set of all structures $\mathcal{A}$ such that $|\mathcal{A}| = k$. For any of the (finite number of) n-place predicate symbols of structures in the set, there are 2^{nk} possible n-ary relations. Similarly, for every n-place function symbol and every constant symbol there are finitely many possible interpretations in a structure of size k. Thus there are only a finite number of structures $\mathcal{A} \in \mathcal{F}_k$.

Let $\mathcal{F} = \bigcup_{k}^{\infty} \mathcal{F}_k$ be the set of all finite structures. Since the union of a countable number of finite sets is itself countable, there are a countable number of finite structures.

A finite structure can be represented as a string over some finite alphabet, and the set of all possible strings over a finite alphabet is countable, so we can enumerate the set of finite structures by enumerating the set of strings and for each one testing to see if it is the syntactically correct representation of a finite structure.

Let $\sigma_1, \sigma_2, \ldots$ be an enumeration of sentences in Σ and let $\mathcal{A}_1, \mathcal{A}_2, \ldots$ be an enumeration of finite structures. P_1 is an effective procedure that enumerates the sentences of Σ that have a counterexample by testing $\mathcal{A}_j \models \neg\sigma_k$ for every $j + k = 2$, then for every $j + k = 3$, and so on. Whenever we obtain a positive result, we output $\neg\sigma_k$ to the list, since it is modeled by $\mathcal{A}_j$. Any σ that has a counterexample will eventually appear in the list.

P_2 is an effective procedure that enumerates the sentences of Σ that are valid by enumerating the set of valid wffs and outputing only those that belong to Σ.

By submitting any $\sigma \in \Sigma$ to P_1 and P_2 in parallel, we decide the validity of σ, since eventually one of the two algorithms must produce a positive result in a finite number of steps.

Problem 2.5.6

Let $\Gamma = \{\neg\forall v_1 P v_1, P v_1, P v_2, P v_3, \ldots\}$. Consider a structure $\mathcal{A}$ with $a, b \in \mid \mathcal{A} \mid$ such that $a \in P^{\mathcal{A}}$ and $b \notin P^{\mathcal{A}}$.

Define the valuation $s : V \rightarrow \mid \mathcal{A} \mid$ such that $s(v_i) = a$ for all i. Then $(\mathcal{A}, s) \models \Gamma$, so that Γ is satisfiable. By Soundness (Corollary 25E) Γ is also consistent.

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Solution to **2.5.6** *by Assaf Kfoury*
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