BU CLA MA 531 Computability and Logic
Fall 1995

Solutions for Problem Set 4

Problem 2.2.19a

Let $\alpha = \exists x_1 \ldots \exists x_k \forall x_{k+1} \ldots \forall x_{k+n} \theta$ be a sentence where θ is quantifier-free and contains no function symbols, and suppose $\mathcal{A} \models \alpha$. Then–

$$(\mathcal{A}, s(x_1|d_1, \ldots, x_k|d_k) \models \forall x_{k+1} \ldots \forall x_{k+n} \theta \quad \text{for some } d_1 \ldots d_k \in \mid \mathcal{A} \mid \text{ and } s : V \rightarrow \mid \mathcal{A} \mid.$$

Let $\mathcal{B}$ be a substructure of $\mathcal{A}$ defined as follows:

(1) $\mid \mathcal{B} \mid = \{d_1, \ldots, d_k\}$ (Actually any finite superset of $\{d_1, \ldots, d_k\}$ will suffice.)

(2) The predicate symbols and constant symbols of $\mathcal{B}$ are exactly those of $\mathcal{A}$ restricted to $\mid \mathcal{B} \mid$.

(3) $\mathcal{B}$ has no function symbols.

$\mathcal{B}$ is finite and $(\mathcal{B}, s(x_1|d_1, \ldots, x_k|d_k)) \models \forall x_{k+1} \ldots \forall x_{k+m} \theta$ for every $s : V \rightarrow \mid \mathcal{B} \mid$. (A detailed proof of this last fact is by induction on universal wffs.) Hence $\mathcal{B} \models \exists x_1 \ldots \exists x_k \forall x_{k+1} \ldots \forall x_{k+m} \theta$; that is, $\mathcal{B} \models \alpha$.

Problem 2.2.19b

Let $\mathcal{B}$ and α be the structure and arbitrary $\exists_2$ sentence defined above in (**a**). We choose a point d (outside $\mid \mathcal{B} \mid$) that is not P-related to any point in $\mid \mathcal{B} \mid$. Then we define $\mathcal{B}_1$ exactly as $\mathcal{B}$ except that $\mid \mathcal{B}_1 \mid = \mid \mathcal{B} \mid \bigcup \{d\}$.

Now clearly α is still true in $\mathcal{B}_1$, yet $\mathcal{B}_1 \not\models \forall x \exists y Pxy$, since we obtain a counterexample by binding d to the variable x.

Problem 2.5.5b

Let Σ be the set of all $\forall_2$ sentences without function symbols. We will show that for any $\sigma \in \Sigma$, if $\neg\sigma$ is modeled by any structure then it is modeled by a structure having a finite universe. This fact, combined with the result of **2.5.5a**, will prove that the set of $\forall_2$ sentences is decidable.

Let $\psi = \neg\sigma$. That is, $\psi = \neg(\forall x_1 \ldots \forall x_k \exists x_{k+1} \ldots \exists x_{k+n} \theta)$, where θ is quantifier-free and contains no function symbols.

Then equivalently $\psi = \exists x_{k+1} \ldots \exists x_{k+n} \forall x_1 \ldots \forall x_k \neg\theta$ (since $\exists x \alpha$ is an abbreviation for $\neg\forall x \neg\alpha$ and $\neg\neg\alpha$ is logically equivalent to α).

Thus $\neg\theta$ is an $\exists_2$ sentence, and we already know from **2.2.19a** that if an $\exists_2$ sentence is true in a structure $\mathcal{A}$, then it is true in a finite substructure of $\mathcal{A}$.

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