BU CLA MA 531 Computability and Logic

Fall 1995

Solutions for Problem Sets 5 and 6

Problem 2.6.2a

$\Sigma_1 = \emptyset \quad \subseteq \quad \Sigma_2 = \{\ \sigma : \sigma \text{ is a sentence }\}$
$\text{Mod}\Sigma_2 = \emptyset \quad \subseteq \quad \text{Mod}\Sigma_1 = \{\mathcal{A} : \mathcal{A} \text{ is a structure}\}$

$\mathcal{K}_1 = \emptyset \quad \subseteq \quad \mathcal{K}_2 = \{\Re\}$, where $\Re$ is the field of reals.
$\text{Th } \mathcal{K}_2 = \{\sigma : \sigma \text{ is true in the real field }\} \quad \subseteq \quad \{\sigma : \sigma \text{ is valid }\} = \text{Th } \mathcal{K}_1$.

Problem 2.6.2b

$\Sigma \subseteq \text{Th Mod } \Sigma$ is trivially true. If $\Sigma = \{\forall x Px,\ \forall x Px \rightarrow \forall x Qx\}$, then $\forall x Qx \in \text{Th Mod } \Sigma$, so Σ is not identical to Th Mod Σ, thus the $\subseteq$ relationship.

Again, $\mathcal{K} \in \text{Mod Th } \mathcal{K}$ is obvious. Let $\mathcal{K}$ be the class of all finite structures $\mathcal{A}$ such that $\mathcal{A} \models \varphi$, where φ is the conjunction of the axioms for total ordering (page 149). The sentence φ has an infinite model $\mathcal{B}$. Thus $\mathcal{B} \in \text{Mod Th } \mathcal{K}$ and $\mathcal{K}$ is not identical to Mod Th $\mathcal{K}$. Conclude that $\mathcal{K} \subseteq$ Mod Th $\mathcal{K}$.

Problem 2.6.6

We are given a structure $\mathcal{A} = (\mathcal{N}, <)$.
Let $\mathcal{B} = (B, <^{\mathcal{B}})$, where $B = \{\ (0, m)\ :\ m \in \mathcal{N}\ \} \bigcup\ \{(1, m)\ :\ m \in \mathcal{Z}\}$, and–

$$\begin{aligned} (a, m) <^B (b, n) \quad \text{iff} \quad & a = 0 \text{ and } b = 1, \text{ or} \\ & a = b = 0 \text{ and } m <^A n, \text{ or} \\ & a = b = 1 \text{ and } m <^B n. \end{aligned}$$

We must show that $\mathcal{A} \equiv \mathcal{B}$.
Consider $\mathcal{A}' = (\mathcal{A}, S^{\mathcal{A}}, 0^{\mathcal{A}})$ and $\mathcal{B}' = (\mathcal{B}, S^{\mathcal{B}}, 0^{\mathcal{B}})$.
By the set of axioms A_L of the theory of $(\mathcal{N}, 0, S, <)$, we have $\mathcal{A}' \models A_L$ and $\mathcal{B}' \models A_L$. Therefore, Cn $A_L \subseteq Th\ \mathcal{A}'$ and Cn $A_L \subseteq Th\ \mathcal{B}'$
By Corollary 32B, Cn A_L is complete. Then–

$$\text{Th } \mathcal{A}' = \text{Cn } A_L = \text{Th } \mathcal{B}'$$
$$\mathcal{A}' \equiv \mathcal{B}'$$
$$\mathcal{A} \equiv \mathcal{B}$$

Problem 2.6.9

We will prove the contrapositive.

Assume that there is *not* a finite number k such that σ is true in all models $\mathcal{A}$ of T for which $|\mathcal{A}| \geq k$. Thus $\Sigma = T \bigcup \{\neg\sigma\}$ has arbitrarily large finite models, and by Theorem 26B an infinite model $\mathcal{B}$.

Conclude that σ is not true in all infinite models of T. In particular, $\mathcal{B} \not\models \sigma$.

Problem 2.7.2

Let π be an interpretation of L_0 into T_1 defined as follows:

$\pi_\forall$ is the identify function.

$\pi_+(x, y, z) = P^+(x, y, z)$

$\pi_\bullet(x, y, z) = P^\bullet(x, y, z)$

Now let φ be an L_0-formula that defines some relation. We will give an L_1-formula φ^π that defines the same relation. The definition of φ^π is by induction on wffs, and in case φ is atomic, by induction on the number of instances of a function symbol in φ.

First, suppose that φ is atomic. Because L_0 has no predicate symbols, φ must be of the form $(t_1 \approx t_2)$, where t_1 and t_2 are expressions generated by the variables and function symbols. If there are no function symbols, then $\varphi = (x_1 \approx x_2)$ for variables x_1 and x_2, and we have $\varphi^\pi = (x_1 \approx x_2)$.

If there are function symbols in $(t_1 \approx t_2)$, then replace the rightmost instance of $f x_1 x_2$ with a new variable y, obtaining a formula φ_y. Then $\varphi^\pi = \forall y(\pi_f(x_1, x_2, y) \rightarrow (\varphi_y)^\pi)$.

For example, if $\varphi = (x_1 + x_2 \approx x_3 \cdot x_4)$, then $\varphi^\pi = \forall y(P^\bullet(x_3, x_4, y) \rightarrow (x_1 + x_2 \approx y)^\pi)$, which is $\forall y(P^\bullet(x_3, x_4, y) \rightarrow \forall z(P^+(x_1, x_2, z) \rightarrow z \approx y))$.

If φ is not atomic, then its interpretation is obvious:

$(\neg\varphi)^\pi = (\neg\varphi^\pi)$

$(\varphi \rightarrow \psi)^\pi = \varphi^\pi \rightarrow \psi^\pi$.

$(\forall x \varphi)^\pi = \forall x \varphi^\pi$ (because $\pi_\forall$ is identity)

Conclude with the help of Lemma 27B that φ^π defines the same relation in $\mathcal{N}_1$ as φ defines in $\mathcal{N}_0$.

Drue Coles
*(**2.6.6** by A.J. Kfoury)*
Created: 95.10.08