BU CLA MA 531 Computability and Logic

Fall 1995

Solutions for Problem Set 7

Problem 3.1.4

Let $\varphi(x)$ be a formula that defines a subset of $\mathcal{N}$. Since $(\mathcal{N}, 0, S)$ admits quantifier elimination, there is a quantifier-free $\psi(x)$ that is logically equivalent to $\varphi(x)$.

Terms of the language are of the form $S^m x$ or $S^m 0$. Atomic formulas are equations. Put ψ(x) in DNF and consider the 8 possible forms of a literal α in $\psi(x)$:

(1) $\alpha = (S^m 0 \approx S^n 0)$. If $m{=}n$, then the interpretation of α is trivially true for every valuation of x. Thus α defines $\mathcal{N}$, which is cofinite. If $m{\neq}n$, then α defines $\emptyset$, which is finite.

(2) $\alpha = \neg(S^m 0 \approx S^n 0)$. Symmetrical to (1). α defines a finite or cofinite set, depending upon whether $m{=}n$ or $m{\neq}n$ respectively.

(3) $\alpha = (S^m x \approx S^n x)$. If $m{=}n$, then by the one-to-one property of successor the interpretation of α is true for any valuation of x. Thus α defines $\mathcal{N}$, which is cofinite. If $m{\neq}n$, then α defines $\emptyset$, which is finite.

(4) $\alpha = \neg(S^m x \approx S^n x)$. Symmetrical to (3).

(5) $\alpha = (S^m x \approx S^n 0)$. An interpretation of α is true for a unique valuation of x if $m{\leq}n$, and for no valuation of x if $m > n$. Thus α defines a singleton or $\emptyset$, both of which are finite.

(6) $\alpha = \neg(S^m x \approx S^n 0)$. Either α defines $\mathcal{N}$ or it defines $\mathcal{N} - \{d\}$ for a unique $d \in \mathcal{N}$. Thus α defines a cofinite set.

(7) $\alpha = (S^m 0 \approx S^n x)$. Identical to (5).

(8) $\alpha = \neg(S^m 0 \approx S^n x)$. Identical to (6).

Conclude that each literal in $\psi(x)$ defines either a finite or a cofinite set.

The class of all finite and cofinite sets is closed under intersection and union. Thus each clause of $\psi(x)$ defines a cofinite set (if all literals define cofinite sets) or a finite set (if at least one literal defines a finite set). Then $\psi(x)$ also defines a finite or cofinite set.

Conversely, let D_1 and D_2 be resepectively a finite and cofinite set of natural numbers. D_1 is definable in $(\mathcal{N}, 0, S)$ by $\varphi = (v_1 \approx S^{a_0} 0 \vee \ \ldots \ \vee v_1 \approx S^{a_n} 0)$, where the a_i's are the elements of D_1. D_2 is defined by $\psi = (v_1 \not\approx S^{b_0} 0 \wedge \ \ldots \ \wedge v_1 \not\approx S^{b_n} 0)$, where the b_i's are the elements in $\mathcal{N} - D_2$.

Problem 3.1.5

By way of proving a contradiction, suppose the ordering relation $R = \{ < m, n > \ : \ m < n\}$ is definable in $\Re_s$ by the wff $\varphi(x, y)$. This means–

$$R = \{ < m, n > \ : \ \Re_s \models \ \varphi(S^m 0, S^n 0) \ \}$$

Because Th $\Re_s$ admits quantifier elimination (Theorem 31G), there is a quantifier-free wff $\psi(x, y)$ such that Th $\Re_s \ \models \ \varphi \leftrightarrow \psi$. Therefore ψ also defines R in $\Re_s$. Put ψ in DNF; that is, $\psi = \psi_1 \vee \ \ldots \ \vee \ \psi_r$, where $\psi_p = \alpha_{p1} \wedge \ \ldots \ \wedge \ \alpha_{pq}$ for $p = 1, \ \ldots, r$, and where α_{ij} is a literal with no free variables other than x and y. (With no loss of generality, we can take the wff's $\psi_1, \ \ldots, \psi_r$ to all be conjunctions of the same number of literals.)

Define a new wff $\tilde{\psi}(x)$ obtained from $\psi(x, y)$ by substituting the term $S^{r+1}0$ for y. We now have–

$$\tilde{\psi} = \tilde{\psi}_1 \vee \ \ldots \ \vee \ \tilde{\psi}_r, \text{ where } \tilde{\psi}_p(x) = \psi(x, S^{r+1}0).$$

By **(3.1.4)**, $\tilde{\psi}$ defines a finite or cofinite set in $\Re_s$.

Case 1: Suppose $\tilde{\psi}$ defines a cofinite set. Then for arbitrarily large $m \in \mathcal{N}$ we have $\Re_s \models \tilde{\psi}(S^m 0)$. That is, $\Re_s \models \ \varphi(S^m 0, S^{r+1}0)$. But this contradicts the fact that for only finitely many $m \in \mathcal{N}$ we have $\Re_s \models \ \psi(S^m 0, S^{r+1}0)$.

Case 2: Suppose $\tilde{\psi}$ defines a finite set. Hence, each of $\tilde{\psi}_1, \ \ldots, \ \tilde{\psi}_r$ defines a finite set. In fact, by **(3.1.4)**, each of $\tilde{\psi}_1, \ \ldots, \ \tilde{\psi}_r$ defines either the empty set or exactly one element. Hence, if the set defined by $\tilde{\psi}$ is $\{m_1, \ \ldots \ m_k\}$, then $k \leq r$, and also $\Re_s \models \ \tilde{\psi}(S^{m_i}0)$ or, equivalently, $\Re_s \models \ \psi(S^{m_i}0, S^{r+1}0)$ for $i = 1, \ \ldots, k$. But this contradicts the fact that there are $r + 1$ elements $m \in \mathcal{N}$ such that $\Re_s \models \ \psi(S^m 0, S^{r+1}0)$.

Problem 3.1.6

We know by Corollary 31D that Th $\Re_s$ = Cn A_s, so it suffices to show that Cn A_s is not finitely axiomatizable.

By way of proving a contradiction, assume Cn A_s is finitely axiomatizable. By Theorem 26E, there is a finite $\Sigma \subseteq A_s$ such that Cn Σ = Cn A_s. With no loss of generality, there is a fixed $m \geq 1$ such that:

$$\Sigma = \{ \ S1, S2, S3, S4, \ \ldots, S4.m \ \}$$

(It may be that Σ contains fewer than these axioms, but there is no harm in including them all, as Cn Σ = Cn A_s.)

Let $\mathcal{A} = (A, 0^A, S^A)$ be an uncountable model of A_s. Because $\Sigma \subseteq A_s$, $\mathcal{A}$ is also a model of Σ. Let $\aleph_1$ be the cardinality of $\mathcal{A}$.

We construct another model $\mathcal{B} = (B, 0^B, S^B)$ for Σ, which also has cardinality $\aleph_1$. Let $b_0, \ \ldots, b_n$ be $(n + 1)$ distinct fresh constants, and define–

$B = A \bigcup \{ b_0, \ldots, b_n \}$
$0^B = 0^A$
S^B restricted to $A = S^A$, and $S^B b_0 = b_1,\ S^B b_1 = b_2,\ \ldots, S^B b_n = b_0$.

By inspection, it is clear that $\mathcal{B} \models \Sigma$. But $\mathcal{A}$ and $\mathcal{B}$ are two non-isomorphic models of the same uncountable cardinality $\aleph_1$, contradicting the fact that Cn $\Sigma =$ Cn A_s is $\aleph_1$ categorical, by Theorem 31B.

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