MA531 Homework 8

Ian C. Westmacott

November 8, 1995

1 3.2.1

Let $D \subseteq N$ be eventually periodic. Then there exists positive numbers M and p such that for all $n > M$, $n \in D$ iff $n + p \in D$, by definition. For each $d_i \in D$, $1 < i < m$, less than M construct the formula φ_i:

$$v_i \approx S^{d_i}0$$

There are finitely many such.

Def: A *generating point* of an eventually periodic set D is a natural number q such that $q + k \cdot p \in D$ for all $k \geq 0$ and $q - p \notin D$.

For each generating point d_i of D, construct the formula $\psi_i(v_1)$:

$$\exists v_2(v_1 \approx d_i + \underbrace{v_2 + \ldots + v_2}_{p})$$

There are at most p such.

Let θ be the disjunction of all the φ_is and ψ_is. Then θ defines D. $\square$

2 3.2.4

By theorem 31F, we need only consider formulas of the form:

$$\theta = \exists x(\alpha_0 \wedge \ldots \wedge \alpha_n) \tag{1}$$

where each α_i is an atomic formula or the negation of an atomic formula. The terms of the language consist only of variables, so atomic formulas have the form $x \approx y$ or $x < y$.

We can assume that x appears in each α_i, for if x does not appear in α we have:

$$\exists x(\alpha \wedge \beta) \models \alpha \wedge \exists x\beta$$
$$\alpha \wedge \exists x\beta \models \exists x(\alpha \wedge \beta)$$

Further, we can assume that x appears on only one side of the relation symbol, as $x \approx x$ can be replaced by $\mathbf{T}$, where $\mathbf{T}$ is a closed wff which is always true, and $x < x$ can be replaced by $\mathbf{F}$, where $\mathbf{F}$ is a closed wff which is always false, then these can be removed from the scope of the quantifier as above.

To eliminate the negation symbol, note that $\neg(a \approx b)$ can be replaced by $a < b \vee b < a$, and $\neg(a < b)$ can be replaced by $a \approx b \vee b < a$. By regrouping the atomic formulas and noting that

$$\exists x(\varphi \vee \psi) \models \exists x \varphi \vee \exists x \psi$$
$$\exists x \varphi \vee \exists x \psi \models \exists x(\varphi \vee \psi)$$

we may again reach formulas of the form (1), where now each α_i is atomic, and in which x appears exactly once. Thus θ contains at least one free variable.

Consider the case of one free variable v. Then each α_i is one of:

$$x \approx v \tag{2}$$
$$x < v \tag{3}$$
$$v < x \tag{4}$$

if θ contains more than one of these, then θ violates the axioms of dense linear orderings, and so θ can be replaced by $\mathbf{F}$. In fact, θ must be exactly $\exists x \alpha$, where α is one of (2-4) (since redundent formulas can be removed). In each case, θ can be replaced by $\mathbf{T}$, by the axioms of dense linear orderings.

In the case of two free variables, v_1 and v_2, for each variable we have the same conditions as stated above, so θ is of the form $\exists x(\alpha_1 \wedge \alpha_2)$, where α_i is one of (2-4), and contains v_i. Otherwise θ is false and can be replaced by $\mathbf{F}$.

Case i: $\theta = \exists x(x \approx v_1 \wedge x < v_2)$. Replace θ by $v_1 < v_2$.

Case ii: $\theta = \exists x(x \approx v_1 \wedge v_2 < x)$. Replace θ by $v_2 < v_1$.

Case iii: $\theta = \exists x(x < v_1 \wedge v_2 < x)$. Replace θ by $v_2 < v_1$.

Case iv: $\theta = \exists x(x < v_1 \wedge x < v_2)$. Replace θ by $\mathbf{T}$.

The symmetric cases are similar.

For three free variables, the interesting case is $\theta = \exists x(x \approx v_1 \wedge x < v_2 \wedge v_3 < x)$. In this case replace θ by $v_3 < v_1 \wedge v_2 < v_3$. The other possibility is that θ is of the form $\exists x(x < v_1 \wedge x < v_2 \wedge v_3 < x)$, which can be replaced by $v_3 < v_1 \wedge v_3 < v_2$. As before, if θ contains more than one of (2-4) for the same variable, θ is false and can be replaced by $\mathbf{F}$.