BU CLA MA 531 Computability and Logic

Fall 1995

Solutions for Problem Set 8

Problem 3.2.1

A set D of natural numbers is periodic if there is a positive number p such that for all n, n is in D iff $n+p$ is in D. Let $\{a_1, \ldots, a_n\}$ be the elements of D that are less than p.

D is definable in $(\mathcal{N}, 0, S, <, +)$ by–

$$\varphi = \exists v_2(v_1 \approx S^{a_1}0 + \underbrace{v_2 + \ldots + v_2}_{p \text{ times}}) \vee \ldots \vee \exists v_2(v_1 \approx S^{a_n}0 + \underbrace{v_2 + \ldots + v_2}_{p \text{ times}})$$

Let the eventually periodic set $D' = (D - X) \bigcup Y$, where D is periodic and defined by φ, and X, Y are finite sets whose elements are $\{a_0, \ \ldots, a_m\}$ and $\{b_0, \ \ldots, b_n\}$ respectively. D' is definable in $(\mathcal{N}, 0, S, <, +)$ by–

$$\psi = (\varphi \ \vee (v_1 \approx S^{b_0}0 \vee \ \ldots \ \vee v_1 \approx S^{b_n}0)) \wedge (v_1 \not\approx S^{a_0}0 \wedge \ \ldots \ \wedge v_1 \not\approx S^{a_m}0)$$

Problem 3.2.4

Let $\varphi = \exists x(\alpha_0 \wedge \ \ldots \ \wedge \alpha_n)$. Each α_i is a literal. We will construct a logically equivalent and quantifier-free formula ψ. This will suffice by Theorem 31F to show that the theory of the language admits quantifier elimination.

Terms of the language are variables. Assume–

(1) There are no logically incompatible literals in φ, for otherwise let $\psi = v_1 < v_1$. (A literal and its negation are logically incompatible.)

(2) x occurs free in each α_i, for if not then $\varphi = \alpha_i \wedge \exists x(\alpha_1 \wedge \ \ldots \ \alpha_{i-1} \wedge \alpha_{i+1} \wedge \ldots \ \wedge \alpha_n)$.

(3) A free variable occurs in each literal, for if α_i is $x < x$ or $x \not\approx x$, then (1) applies; and if α_i is $x \approx x$ or $x \not< x$, then it may be removed from φ.

(4) x appears to the left of the predicate symbol, for $v_1 < x$ may be replaced by $(x \not< v_1 \wedge x \not\approx v_1)$.

Thus all literals of φ are in $S = \{\ (x \approx v_j),\ (x \not\approx v_k),\ (x < v_l),\ (x \not< v_m)\ \}$.

Consider $\varphi(v_1)$. We have $j, k, l, m = 1$, and $\varphi(v_1)$ can contain at most two literals in S. Any such pair is satisfiable by every valuation (as is any single literal), so let $\psi = v_1 \approx v_1$.

Now consider $\varphi(v_1, v_2)$. Assume that none of the literals is an equation, for if $\alpha_i = (x \approx v_j)$ then we obtain a logically equivalent formula with either one or no free variables in $\varphi^x_{v_j}$.

Assume also that $\varphi(v_1, v_2)$ contains no negated equalities, for then $\varphi(v_1, v_2)$ is satisfiable by every valuation. To see this, consider $\alpha_i = (x \not\approx v_1)$. We may always find an x other than v_1 to satisfy the remaining literals ($x < v_1$ or $x < v_2$), regardless of the values assigned to v_1 and v_2.

Finally, if $\varphi(v_1, v_2)$ contains only $x < v_1$ or $x < v_2$ or both, then the formula is again satisfied by every valuation. Let $\psi = (v_1 \approx v_2)$.

Lastly, consider $\varphi(v_1, v_2, v_3)$. Assume as above that the formula contains no equalities or negated equalities; that is, the formula contains any combination of $x < v_1$ and $x < v_2$ and $x < v_3$. Any such combination is satisfiable by every valuation. Let $\psi = (v_1 \approx v_1)$.

The construction of ψ can readily be extended to $\varphi(v_1, \ldots, v_n)$.

Drue Coles
Created: 95.11.04