MA531 Homework 9

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1 3.3.1

Let $\mathcal{N} = (N, \cdot, E)$. In $\mathcal{N}$ we can define:

$$\begin{array}{rll} \{0\} & by & \forall x(v_1 \cdot x \approx v_1) \\ < & by & \forall x \exists y(y \not\approx 0 \wedge (xEv_1) \cdot (xEy) \approx xEv_2) \\ \{< n, S(n) >: n \in N\} & by & \forall x((xEv_1) \cdot x \approx xEv_2) \\ \{< m, n, m+n >: m, n \in N\} & by & \forall x((xEv_1) \cdot (xEv_2) \approx xEv_3) \end{array}$$

□

2 3.3.2

Let τ be a quantifier-free sentence in the language of $\mathcal{N}$. Since τ is closed, each term t in τ is such that

$$A_E \vdash t \approx S^n 0 \tag{1}$$

for some n, by Lemma 33B. Thus atomic formulas are of the form:

$$S^n 0 \approx S^m 0 \tag{2}$$

and

$$S^n 0 < S^m 0 \tag{3}$$

Formulas of the form (2) can be deduced from A_E if $n = m$ and are refutable from A_E if $n \neq m$ by repeated use of S2 and S1. We have

$$S^n 0 \approx S^m 0$$

and S2

$$\forall x \forall y(Sx \approx Sy \to x \approx y)$$

and modus ponens gives us

$$S^{n-1} \approx S^{m-1}$$

after $min\{n, m\}$ iterations, one or both of the terms is $\mathbf{0}$. If both, then (2) is tautologically true, if only one, then S1 refutes it. Formulas of the form (3) can be deduced from A_E if $n < m$ and refuted if $n \geq m$. We have

$$S^n 0 < S^m$$

and L1

$$\forall x \forall y (x < Sy \leftrightarrow (x < y \vee x \approx y))$$

(recall that $x \leq y$ abbreviates $x < y \vee x \approx y$.) and modus ponens gives us

$$S^n 0 < S^{m-1} 0 \vee S^n 0 \approx S^{m-1} 0$$

After m iterations, we have

$$S^n 0 < 0 \vee S^n 0 \approx S^{m-(m-1)} 0 \vee \ldots \vee S^n 0 \approx S^{m-1} 0$$

the first atomic formula is refuted by L2, and the rest are of the form (2). If any one of these can be deduced, then the whole sentence can be deduced, otherwise it is refuted.

Given two atomic formulas α and β of the form (2) or (3), $\neg\alpha$ is trivially deduced or refuted, and $\alpha \rightarrow \beta$ is refuted if β is refuted and α is deduced. Otherwise it is deduced. $\square$