CS 512, Spring 2018, Handout 06

Linear Temporal Logic (LTL)

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4 February 2018

reading assignment

- [PMC, Section 5.1, pages 229-270] : This is a long chapter, more than 40 pages. Start from the very beginning and focus on the motivation and examples.
- [LCS, Sections 3.1 and 3.2, pages 172-186] : There is considerable overlap with the material in [PMC], from a somewhat different perspective.
- Differences in the syntax of LTL between [PMC] and [LCS]:

modality	where in [PMC]	where in [LCS]
"next"	$\bigcirc$, page 231	X, page 176
"until"	U, page 231	U, page 176
"eventually"	$\Diamond$, page 232	F, page 176
"always"	$\Box$, page 232	G, page 176

- More on differences in the syntax in [PMC, Remark 5.16, page 247].
- We follow notation and conventions of [PMC] rather than [LCS] – except that we use "$\mathbb{U}$" instead of "U" to avoid any possible confusion with set union "$\cup$".

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- We follow notation and conventions of [PMC] rather than [LCS] – except that we use "$\mathbb{U}$" instead of "U" to avoid any possible confusion with set union "$\cup$".
- In the context of temporal logics (*e.g.*, all those considered in [PMC] and those in [LCS, Chap 3]), $\{\Diamond, \Box\}$ are usually called *temporal connectives* or *operators* .

  In the context of modal logics (*e.g.*, those in [LCS, Chap 5]), $\{\Diamond, \Box\}$ are usually called *modal connectives* or *operators* .

  They are very close, but not identical, in the way $\{\Diamond, \Box\}$ are used as temporal and modal connectives.

linear temporal logic (LTL)

- syntax of LTL over the set AP of atomic propositions [PMC, Def. 5.1, p. 231] :

$$\varphi, \psi \ ::= \ \mathtt{true} \mid a \mid \neg\varphi \mid \varphi \wedge \psi \qquad \text{propositional logic}$$

$$\mid \ \bigcirc\varphi \qquad \text{“next } \varphi\text{”}$$

$$\mid \ \varphi \ \mathbin{\mathsf{U}} \ \psi \qquad \text{“}\varphi\text{ until }\psi\text{”}$$

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- syntax of LTL over the set AP, with more connectives [LCS, pp. 175-176] :

$$\begin{array}{rlll} \varphi, \psi & ::= & \mathtt{true} \mid a \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \to \psi \mid \cdots & \text{propositional logic} \\ & \mid & \bigcirc \varphi & \text{“next } \varphi\text{”} \\ & \mid & \varphi \,\mathbb{U}\, \psi & \text{“}\varphi \text{ until } \psi\text{”} \\ & \mid & \Diamond \varphi & \text{“eventually } \varphi\text{”} \\ & \mid & \Box \varphi & \text{“always } \varphi\text{”} \\ & \mid & \cdots & \end{array}$$

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- In [PMC] / [LCS] “$\vee$”, “$\to$”, etc., are *shorthand for* / *equivalent to* combinations of “$\wedge$” and “$\neg$”, and “$\Diamond$”, “$\Box$”, etc., are *shorthand for* / *equivalent to* combinations of “$\mathbb{U}$” and “$\neg$”, as shown in [PMC, p. 232] / [LCS, pp. 184-187] .

linear temporal logic (LTL)

precedence rules to simplify syntax and omit matching parentheses:

- unary connectives, both logical and temporal, bind most tightly,
- binary temporal connectives $\{\mathbb{U}, \ldots\}$ bind more tightly than binary logical connectives $\{\wedge, \vee, \rightarrow, \ldots\}$,
- binary logical connectives $\{\wedge, \vee\}$ bind more tightly than $\{\rightarrow\}$,
- in case of doubt, use matching parentheses.

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intuitive and helpful readings of temporal/modal connectives:

- $\Diamond\varphi$: "eventually φ" (temporal), "possibly φ" (modality), "in some future state"
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example: let $\pi \triangleq s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \cdots$ be an infinite path in a transition system:

- $\Box\Diamond\varphi$: "infinitely often φ,"

  "$\forall i\ \exists j\ (j \geqslant i$ ***and*** φ holds at state s_j)" or also "$\overset{\infty}{\exists} j$ (φ holds at state s_j)"

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- $\Diamond\Box\varphi$: "eventually forever φ,"

  "$\exists i\ \forall j\ (j \geqslant i$ ***implies*** φ holds at state s_j)" or also "$\overset{\infty}{\forall} j$ (φ holds at state s_j)"

formal semantics of LTL

- Satisfaction of LTL formulas is relative to ω-words $\sigma \triangleq A_0 A_1 A_2 \cdots \in (2^{\mathsf{AP}})^\omega$. Using the notation of [PMC, page 235], we define for every $j \geqslant 0$:

$$\sigma[j\ldots] \triangleq A_j A_{j+1} A_{j+2} \cdots \quad \text{(the suffix of } \sigma \text{ starting at } A_j\text{)}$$

- We write $\sigma \models \varphi$ and say "σ satisfies (or models, or makes true) the formula φ"
- Given a fixed $\sigma \triangleq A_0 A_1 A_2 \cdots \in (2^{\mathsf{AP}})^\omega$, satisfaction of LTL formulas φ by σ is defined by induction on φ:

1. $\sigma \models \texttt{true}$
2. $\sigma \models a$ iff $a \in A_0$
3. $\sigma \models \neg\varphi$ iff $\sigma \not\models \varphi$
4. $\sigma \models \varphi \wedge \psi$ iff $\sigma \models \varphi$ and $\sigma \models \psi$

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  4. $\sigma \models \varphi \wedge \psi$ iff $\sigma \models \varphi$ and $\sigma \models \psi$
  5. $\sigma \models \bigcirc\varphi$ iff $\sigma[1\ldots] = A_1 A_2 A_3 \cdots \models \varphi$
  6. $\sigma \models \varphi \mathbin{\mathsf{U}} \psi$ iff there is $j \geqslant 0$ such that $\sigma[j\ldots] \models \psi$
     and $\sigma[i\ldots] \models \varphi$ for every $0 \leqslant i < j$

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