CS 512, Spring 2018, Handout 08

Equivalences of LTL Formulas

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additional temporal operators

- The syntax of LTL according to [LCS, Def 3.1, page 175] includes two additional binary temporal operators:

  W, pronounced "weak until",
  R, pronounced "release".

- Using the notation of Handout 06, the formal semantics of W and R are defined by:

7. $\sigma \models \varphi \,\mathsf{W}\, \psi$ iff **either** $\sigma \models \varphi \,\mathbb{U}\, \psi$
   or $\sigma[k\ldots] \models \varphi$ for every $k \geqslant 0$

8. $\sigma \models \varphi \,\mathsf{R}\, \psi$ iff **either** there is $j \geqslant 0$ such that $\sigma[j\ldots] \models \varphi$ and $\sigma[i\ldots] \models \psi$ for every $0 \leqslant i \leqslant j$
   or $\sigma[k\ldots] \models \psi$ for every $k \geqslant 0$

equivalence between LTL formulas, [LCS, pp 184-186]

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- **Definition** [PMC, Def. 5.17, page 248], [LCS, Def. 3.9, page 184]:
  Formulas φ and ψ of LTL are equivalent, in symbols $\varphi \equiv \psi$, iff
  Words$(\varphi) =$ *Words*(ψ).

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- **Definition** [PMC, Def. 5.17, page 248], [LCS, Def. 3.9, page 184]: Formulas φ and ψ of LTL are equivalent, in symbols $\varphi \equiv \psi$, iff *Words*$(\varphi) =$ *Words*(ψ).

- **Dualities in LTL** :

- $\neg\Box\varphi \equiv \Diamond\neg\varphi$
- $\neg\Diamond\varphi \equiv \Box\neg\varphi$
- $\neg\bigcirc\varphi \equiv \bigcirc\neg\varphi$
- $\neg(\varphi \ \mathbb{U} \ \psi) \equiv \neg\varphi \, \mathsf{R} \, \neg\psi$
- $\neg(\varphi \, \mathsf{R} \, \psi) \equiv \neg\varphi \ \mathbb{U} \ \neg\psi$

- For a rigorous proof of the last equivalence: go to page 8. Rigorous proofs for all other equivalences are similar.

equivalence between LTL formulas, [LCS, pp 184-186]

- **Distributive Laws in LTL** :

equivalence between LTL formulas, [LCS, pp 184-186]

- **Distributive Laws in LTL** :

- $\bigcirc(\varphi \vee \psi) \equiv \bigcirc\varphi \vee \bigcirc\psi$
- $\bigcirc(\varphi \wedge \psi) \equiv \bigcirc\varphi \wedge \bigcirc\psi$
- $\bigcirc(\varphi \; \mathbb{U} \; \psi) \equiv (\bigcirc\varphi) \; \mathbb{U} \; (\bigcirc\psi)$

- $\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$

- $\Box(\varphi \wedge \psi) \equiv \Box\varphi \wedge \Box\psi$

- $\varphi \; \mathbb{U} \; (\psi_1 \vee \psi_2) \equiv (\varphi \; \mathbb{U} \; \psi_1) \vee (\varphi \; \mathbb{U} \; \psi_2)$
- $(\varphi_1 \wedge \varphi_2) \; \mathbb{U} \; \psi \equiv (\varphi_1 \; \mathbb{U} \; \psi) \wedge (\varphi_2 \; \mathbb{U} \; \psi)$

equivalence between LTL formulas, [LCS, pp 184-186]

- **Inter-Definitions in LTL** :

- $\Diamond \varphi \equiv \neg \Box \neg \varphi$
- $\Box \varphi \equiv \neg \Diamond \neg \varphi$
- $\Diamond \varphi \equiv \textit{true} \; \mathbb{U} \; \varphi$
- $\Box \varphi \equiv \textit{false} \; \mathsf{R} \; \varphi$
- $\varphi \; \mathbb{U} \; \psi \equiv \varphi \, \mathsf{W} \, \psi \wedge \Diamond \psi$
- $\varphi \, \mathsf{W} \, \psi \equiv \varphi \; \mathbb{U} \; \psi \vee \Box \varphi$

equivalence between LTL formulas

- **Idempotency Laws in LTL** :
- $\Diamond\Diamond\varphi \equiv \Diamond\varphi$
- $\Box\Box\varphi \equiv \Box\varphi$
- $\varphi \,\mathbb{U}\, \psi \equiv \varphi \,\mathbb{U}\, (\varphi \,\mathbb{U}\, \psi)$

equivalence between LTL formulas

- **Idempotency Laws in LTL** :

- $\Diamond \Diamond \varphi \equiv \Diamond \varphi$
- $\Box \Box \varphi \equiv \Box \varphi$
- $\varphi \, \mathbb{U} \, \psi \equiv \varphi \, \mathbb{U} \, (\varphi \, \mathbb{U} \, \psi)$

- Some (perhaps surprising) equivalences in LTL:

- $\Box \Diamond \Box \varphi \equiv \Diamond \Box \varphi$
- $\Diamond \Box \Diamond \varphi \equiv \Box \Diamond \varphi$
- $\Box (\Diamond \varphi \vee \Diamond \psi) \equiv \Box \Diamond \varphi \vee \Box \Diamond \psi$

equivalence between LTL formulas

- $\Diamond$ has similarities with $\exists$, *e.g.*, $\Diamond\,(\varphi \vee \psi) \;\equiv\; (\Diamond\,\varphi \;\vee\; \Diamond\,\psi)$
- $\Box$ has similarities with $\forall$, *e.g.*, $\Box\,(\varphi \wedge \psi) \;\equiv\; (\Box\,\varphi \;\wedge\; \Box\,\psi)$
- $\Diamond$ does **not** distribute over $\wedge$:

  there is a model/transition system TS that distinguishes $\Diamond\,(\varphi \wedge \psi)$ from $(\Diamond\,\varphi \;\wedge\; \Diamond\,\psi)$ for some φ and ψ

rigorous proof of an equivalence:

To prove the equivalence $\boxed{\neg(\varphi \,\mathsf{R}\, \psi) \equiv \neg\varphi \;\mathbb{U}\; \neg\psi}$ we need to show:

$$\text{for every path } \pi \quad \Big(\pi \models \neg(\varphi \,\mathsf{R}\, \psi) \qquad \text{iff} \qquad \pi \models \neg\varphi \;\mathbb{U}\; \neg\psi \Big)$$

Instead of “for every path” we can equivalently show the bi-implication “for every ω-trace”.

Equivalently, we need to show:

$$\text{for every path } \pi \quad \Big(\pi \models \varphi \,\mathsf{R}\, \psi \qquad \text{iff} \qquad \pi \models \neg(\neg\varphi \;\mathbb{U}\; \neg\psi) \Big)$$

What follows in the succeeding pages is a proof of this equivalence:

Instead of writing “$\pi[i\ldots]$” for the suffix of π starting with its i-th entry, we write “π^i”.

rigorous proof of an equivalence:

For an arbitrarily given path π, we have the following sequence of equivalences:

$$(1) \qquad \pi \models \neg(\neg\varphi \; \mathbb{U} \; \neg\psi) \qquad \text{iff}$$

(by the definition of $\mathbb{U}$)

$$(2) \qquad \neg(\exists j \geqslant 0) \left(\pi^j \models \neg\psi \;\wedge\; (\forall i < j)\,(\pi^i \models \neg\varphi) \right) \qquad \text{iff}$$

(by the semantics of $\neg$)

$$(3) \qquad \neg(\exists j \geqslant 0) \left(\pi^j \not\models \psi \;\wedge\; (\forall i < j)\,(\pi^i \not\models \varphi) \right) \qquad \text{iff}$$

(by the duality of $\exists$ and $\forall$)

$$(4) \qquad (\forall j \geqslant 0)\, \neg \left(\pi^j \not\models \psi \;\wedge\; (\forall i < j)\,(\pi^i \not\models \varphi) \right) \qquad \text{iff}$$

(by de Morgan's law)

$$(5) \qquad (\forall j \geqslant 0) \left(\neg(\pi^j \not\models \psi) \;\vee\; \neg(\forall i < j)\,(\pi^i \not\models \varphi) \right) \qquad \text{iff}$$

(by the semantics of $\neg$ and the duality of $\exists$ and $\forall$)

$$(6) \qquad (\forall j \geqslant 0) \left(\pi^j \models \psi \;\vee\; (\exists i < j)\,(\pi^i \models \varphi) \right)$$

rigorous proof of an equivalence:

$$(6) \qquad (\forall j \geqslant 0) \left(\pi^j \models \psi \ \vee \ (\exists i < j)\,(\pi^i \models \varphi) \right) \qquad \text{iff}$$

(by the duality of $\rightarrow$ and $\vee$)

$$(7) \qquad (\forall j \geqslant 0) \left(\pi^j \not\models \psi \ \rightarrow \ (\exists i < j)\,(\pi^i \models \varphi) \right) \qquad \text{iff}$$

(by a re-arrangement of subexpressions)

$$(8) \qquad (\forall j \geqslant 0)\,(\pi^j \models \psi) \qquad (8.1)$$

$$\textbf{or} \ \ (\exists i \geqslant 0) \left(\pi^i \models \varphi \wedge (\forall k \leqslant i)(\pi^k \models \psi) \right) \qquad (8.2)$$

All the preceding equivalences, from (1) to (7), are straightforward. The one which needs further justification is (8) = $\big((8.1)$ **or** $(8.2)\big)$. We consider two possibilities for the path π:

(a) **Either** for every $j \geqslant 0$, we have $\pi^j \models \psi$, in which case both (7) and (8.1) hold – or, which is easier to see, both (6) and (8.1) hold. Hence, (6), (7) and (8) hold.

(b) **Or** there are $0 \leqslant j_0 < j_1 < j_2 < \cdots$ such that $\pi^{j_0} \not\models \psi, \pi^{j_1} \not\models \psi, \pi^{j_2} \not\models \psi, \ldots$ and for all $k \notin \{j_0, j_1, j_2, \ldots\}$, we have $\pi^k \models \psi$. Hence, if (7) holds, there is $i < j_0$ such that $\pi^i \models \varphi$ and for all $k \leqslant i < j_0$, it holds that $\pi^k \models \psi$, thus implying (8.2). Conversely, if (8.2) holds, then (7) holds. Hence, (7) iff (8).

Hence, whether (a) or (b) is the case, we have (7) iff (8).

rigorous proof of an equivalence:

A closer look at (8) shows that:

$$(8) \qquad (\forall j \geqslant 0)\,(\pi^j \models \psi) \ \textbf{or} \ (\exists i \geqslant 0)\left(\pi^i \models \varphi \wedge (\forall k \leqslant i)(\pi^k \models \psi)\right)$$

is a more formal re-wording of the semantics of R. Hence, (8) holds iff:

$$(9) \qquad \pi \models \varphi \,\mathsf{R}\, \psi$$

Since π is an arbitrarily given path, we conclude that for every path π, we have (1) iff (9). This completes our rigorous proof.

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