CS 512, Spring 2018, Handout 23

Modal Logics

Assaf Kfoury

April 22, 2018 (adjusted April 27, 2018)

syntax [LCS, Section 5.2.1], click **here** for Wikipedia article

$\varphi, \psi ::=$	$\mathtt{true} \mid \mathtt{false} \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \cdots$	propositional logic
$\mid$	$\Box\varphi$	"box φ"
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- "$\Box$" is now called a **modality** or a **modal operator**, rather than a **temporal connective**, and ditto for "$\Diamond$".
- "$\Box\varphi$" can be also read as "necessarily φ", and "$\Diamond\varphi$" as "possibly φ".
- Another intuitive understanding of "$\Box$" and "$\Diamond$":
  - In many ways, "$\Box$" in modal logic is like "$\forall\bigcirc$" (not "$\forall\Box$") in CTL.
  - In many ways, "$\Diamond$" in modal logic is like "$\exists\bigcirc$" in CTL.

  Caution: Although intuitively helpful, since we started with temporal logics before turning to modal logics, this correspondence with CTL does not always hold.

semantics [LCS, Section 5.2.2]

for a detailed and somewhat different presentation of the semantics of modal logic, click **here** for the Wikipedia article on **Kripke semantics**.

[1] $\mathcal{P}(\mathsf{AP})$ is the powerset of AP, which is the same as 2^{AP} in earlier handouts. Note that we can take the labelling function in the alternative form $L : AP \to \mathcal{P}(W)$; the two forms give equivalent definitions of Kripke models.

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- A model $\mathcal{M}$ of **basic modal logic** (also called a **Kripke model**) is a triple:

  $$\mathcal{M} = (W, R, L) \qquad \text{where}$$

  - W is the **set of possible worlds**,
  - $R \subseteq W \times W$ is a binary **accessibility relation**,
  - $L : W \to \mathcal{P}(\mathsf{AP})$ is a **labelling function**[1]

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- In different accounts of modal logic, the members of W may be referred by different names: **possible worlds**, **states**, **nodes**, **times**, etc., depending on the semantics of the logic.

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As usual, the **formal semantics** is **syntax-directed**,
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basis steps

- $\mathcal{M}, x \Vdash$ **`true`**
- $\mathcal{M}, x \nVdash$ **`false`**
- $\mathcal{M}, x \Vdash p \quad$ iff $\quad p \in L(x)$

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induction steps: propositional-logic connectives

- $\mathcal{M}, x \Vdash \neg\varphi$ iff $\mathcal{M}, x \nVdash \varphi$
- $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$
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induction steps: modal-logic connectives

- $\mathcal{M}, x \Vdash \Box\varphi$ iff $\Big(\mathcal{M}, y \Vdash \varphi$ for **every** $y \in W$ such that $R(x, y) \Big)$
- $\mathcal{M}, x \Vdash \Diamond\varphi$ iff $\Big(\mathcal{M}, y \Vdash \varphi$ for **some** $y \in W$ such that $R(x, y) \Big)$

semantics [LCS, Section 5.2.2]

Given a set Γ of WFF's and a WFF φ, all of basic modal logic, we say:

- **Γ (semantically) entails φ**, in symbols $\Gamma \models \varphi$, iff
  for every model $\mathcal{M} = (W, R, L)$ and every world $x \in W$,
  if $\mathcal{M}, x \Vdash \Gamma$ then $\mathcal{M}, x \Vdash \varphi$.

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  Remark: Note the switch from "$\Vdash$" to "$\models$", following [LCS, p. 313], although **not** every account of modal logic follows this convention. Somewhat at odds with other accounts, [LCS, p. 310] pronounces "$\Vdash$" as "satisfies" (rather than "forces"), while *satisfaction* elsewhere is usually strictly reserved to denote a **semantic** relation.

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- **$\mathcal{M}$ satisfies φ** or **φ is true in $\mathcal{M}$**, in symbols $\mathcal{M} \models \varphi$, iff for every world $x \in W$ we have $\mathcal{M}, x \Vdash \varphi$.

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Given a WFF φ of basic modal logic, we say [LCS, p. 314]:

- **φ is (semantically) valid**, in symbols $\models \varphi$, iff
  for every model $\mathcal{M}$ we have $\mathcal{M} \models \varphi$.

useful equivalences

φ and ψ are **(semantically) equivalent**, in symbols $\varphi \equiv \psi$, iff both:

$$\varphi \models \psi \quad \text{and} \quad \psi \models \varphi$$

or, equivalently, iff: $\models \varphi \leftrightarrow \psi$.

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3. $\Box(\varphi \wedge \psi) \;\equiv\; \Box\varphi \wedge \Box\psi$
4. $\Diamond(\varphi \vee \psi) \;\equiv\; \Diamond\varphi \vee \Diamond\psi$

   the two preceding are "distributivity laws" of: "$\Box$ over $\wedge$" and "$\Diamond$ over $\vee$"

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5. $\Box(\varphi \vee \psi) \;\not\equiv\; \Box\varphi \vee \Box\psi$ **but** $\models\; \Box\varphi \vee \Box\psi \;\longrightarrow\; \Box(\varphi \vee \psi)$
6. $\Diamond(\varphi \wedge \psi) \;\not\equiv\; \Diamond\varphi \wedge \Diamond\psi$ **but** $\models\; \Diamond(\varphi \wedge \psi) \;\longrightarrow\; \Diamond\varphi \wedge \Diamond\psi$

   We can also obtain the two preceding from
   the duality of $\Box$ and $\Diamond$ and the duality of $\wedge$ and $\vee$.

useful equivalences

(more **subtle**, see [LCS, bottom of p. 311 and top of p. 312])

7. $\Box\mathtt{true} \equiv \mathtt{true}$
8. $\Diamond\mathtt{false} \equiv \mathtt{false}$
9. $\models\ \mathtt{false} \longrightarrow \Box\mathtt{false}$ **but** $\not\models\ \Box\mathtt{false} \longrightarrow \mathtt{false}$
10. $\models\ \Diamond\mathtt{true} \longrightarrow \mathtt{true}$ **but** $\not\models\ \mathtt{true} \longrightarrow \Diamond\mathtt{true}$

useful equivalences

11. More complicated equivalences can be obtained from appropriate substitutions into equivalences of propositional logic. For example, we know that $(p \to \neg q) \equiv \neg(p \wedge q)$. So, if we substitute $\Box\varphi \wedge (\psi \to \varphi)$ for p and $\theta \to \Diamond(\psi \vee \varphi)$ for q, we obtain the following equivalence:

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12. Every equivalence can be turned into a (semantic) validity, and vice-versa. For example, the equivalence $\Box\varphi \equiv \neg\Diamond\neg\varphi$ can be turned into the semantically valid WFF $(\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi)$.

    Remark: *Equivalence* denoted by "$\equiv$" is a notion at the **meta level**, whereas the symbol "$\leftrightarrow$" is at the **object level**. Thus, $\Box\varphi \equiv \neg\Diamond\neg\varphi$ is **not** a WFF.

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13. More non-trivial equivalences are in [LCS, pp. 312-315] and the exercises for [LCS, Sect 5.2, pp.350-351].

more on duality in modal logic (not in [LCS])

Dualities of all sorts are found in **formal methods** and **mathematical logic**:

- **syntax/proof theory** versus **semantics/model theory**,

  deducibility versus **semantic validity**,

  consistency versus **satisfiability**,

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- between **propositional connectives** (*e.g.*, $\wedge$ and $\vee$), via negation
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- between **modal operators** (*e.g.*, $\Box$ and $\Diamond$), via negation

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- between **modal operators** (*e.g.*, $\Box$ and $\Diamond$), via negation

More generally, two n-ary modal operators Δ and ∇ are dual of each other iff:

$$\Delta(\varphi_1, \ldots, \varphi_n) \equiv \neg\nabla(\neg\varphi_1, \ldots, \neg\varphi_n)$$

Remark: "∇" is pronounced "nabla".

more on duality in modal logic (not in [LCS])

- In the presence of general modal operators, for each n-ary modal operator Δ and its dual ∇, we need to introduce an $(n+1)$-accessibility relation.

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- **Example:** Let Δ_i and ∇_i be dual **2-ary modal operators**, for $i = 1, 2$.
  - **Syntax** is now modified as:

$$\begin{aligned} \varphi, \psi \ ::= \ & \mathbf{false} \mid \mathbf{true} \mid \neg\varphi \mid \varphi \wedge \psi \mid \cdots \mid \qquad \text{(propositional connectives)} \\ & \Delta_1(\varphi, \psi) \mid \Delta_2(\varphi, \psi) \mid \nabla_1(\varphi, \psi) \mid \nabla_2(\varphi, \psi) \end{aligned}$$

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  - A **model** is now a tuple $\mathcal{M} = (W, R_1, R_2, L)$, where R_i is a **3-ary accessibility relation** for 2-ary modal operators Δ_i and ∇_i, $i = 1, 2$.

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  - **Formal semantics** is modified as, for $i = 1, 2$:
    - $\mathcal{M}, x \Vdash \Delta_i(\varphi_1, \varphi_2)$ iff **there are** $y_1, y_2 \in W$ such that $R_i(x, y_1, y_2)$ and $\mathcal{M}, y_1 \Vdash \varphi_1$ and $\mathcal{M}, y_2 \Vdash \varphi_2$
    - $\mathcal{M}, x \Vdash \nabla_i(\varphi_1, \varphi_2)$ iff **for all** $y_1, y_2 \in W$ if $R_i(x, y_1, y_2)$ then $\mathcal{M}, y_1 \Vdash \varphi_1$ and $\mathcal{M}, y_2 \Vdash \varphi_2$

more on duality in modal logic (not in [LCS])

Example: Suppose $\Diamond_i$ and $\Box_i$ are unary modal operators which are dual to each other, for $i = 1, 2$. Examples of specific models for the corresponding modal logic are – actually these are **frames** rather than **models** due to the absence of labelling functions:

- $\mathcal{N} = (\mathbb{N}, S_1, S_2)$ where $\mathbb{N}$ is the set of natural numbers and

$$m\, S_1\, n \quad \text{iff} \quad n = m + 1$$

$$m\, S_2\, n \quad \text{iff} \quad m < n$$

- $\mathcal{B} = (\mathbb{B}, R_1, R_2)$ where $\mathbb{B}$ is the set $\{0, 1\}^*$ of all finite binary strings and

$$s\, R_1\, t \quad \text{iff} \quad t = s0 \;\text{ or }\; t = s1$$

$$s\, R_2\, t \quad \text{iff} \quad s \text{ is a proper prefix of } t$$

more on duality in modal logic (not in [LCS])

Exercise: Which of the following WFF's are satisfied by the frames $\mathcal{N}$ and $\mathcal{B}$ on the preceding slide? Justify your answers:

1. $(\Diamond_1 \varphi \wedge \Diamond_1 \psi) \to \Diamond_1 (\varphi \wedge \psi)$
2. $(\Diamond_2 \varphi \wedge \Diamond_2 \psi) \to \Diamond_2 (\varphi \wedge \psi)$
3. $(\Diamond_1 \varphi \wedge \Diamond_1 \psi \wedge \Diamond_1 \theta) \to \Diamond_1 (\varphi \wedge \psi) \vee \Diamond_1 (\varphi \wedge \theta) \vee \Diamond_1 (\psi \wedge \theta)$
4. $\varphi \to \Diamond_1 \Box_2 \varphi$
5. $\varphi \to \Diamond_2 \Box_1 \varphi$
6. $\varphi \to \Box_1 \Diamond_2 \varphi$
7. $\varphi \to \Box_2 \Diamond_1 \varphi$

axioms in modal logics

[LCS, Section 5.3, pp. 316-328]

formal provability (natural deduction)

[LCS, Sect 5.4, pp. 328-331]

more examples: reasoning in multi-agent systems

[LCS, Sect 5.5, pp. 331-349]

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