BU CLA MA 531: Computability and Logic
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Handout 1

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Hilbert-Style Proof Systems

A deductive calculus such as the one given in Enderton's book, Section 2.4, is often called a *Hilbert-style proof* (or *axiomatic*) *system* — or more simply, a *Hilbert system*. But there are many variations of Hilbert systems, each defined by a collection of *axiom schemes* and a collection of *inference rules*. It is a matter of taste and convenience how we choose axiom schemes and inference rules (so that the resulting system is both sound and complete).

The system in Enderton's book, as well as all the systems for first-order logic except for one (System F) which we list below, include among the logical axioms "all generalizations of tautologies" or "all closed tautologies" or "all tautologies". A *tautology* means a substitution instance of a propositional (or sentential) tautology, just as it is defined in Enderton's, page 106. Propositional tautologies in Enderton's are defined semantically, page 34. But it is also possible to set up a Hilbert system for propositional logic, which derives exactly all the propositional tautologies, i.e. so that the resulting proof system is "sound" and "complete". Enderton does not present any deductive calculus for propositional logic and, therefore, he does not need to prove any soundness and completeness result for it; he restricts his presentation of propositional logic to its semantics.

A Hilbert system for propositional logic

α, β and γ denote arbitrary propositional wff's.

- *Axiom schemes*
  - $(\alpha \to (\beta \to \alpha))$;
  - $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$;
  - $(\neg\beta \to \neg\alpha) \to ((\neg\beta \to \alpha) \to \beta)$.
- *Inference rules*
  - Modus Ponens: $\dfrac{\alpha \qquad \alpha \to \beta}{\beta}$.

The preceding proof system for propositional logic can be found in many texts, for example in [4]. It is not the only Hilbert system for propositional logic; others are given in Sections 1.10, 1.14, and 1.15 in [1], and in Section 19 in [3].[1]

[1] Although the Hilbert system for propostional logic in [3] has relatively many axiom schemes, a total of ten, it has the advantage that by omitting just one of them (corresponding to the law of "excluded middle") the result is a Hilbert system for intuitionistic propositional logic – more on this in later handouts.

We next list a few alternative formulations of Hilbert-style proof systems for first-order logic. We start with a system which is very close to the system in Enderton's book.

System A

Use the following convention: $\varphi(x)$ denotes any wff which has no free variable other than x, and $\varphi(c)$ denotes the result of substituting the constant symbol c for every free occurrence of x in $\varphi(x)$. α, ψ and θ denote arbitrary wff's (satisfying whatever restrictions are added).

- *Axiom schemes*

  A1 All closed tautologies;

  A2 $(\forall x\ \varphi(x)) \to \varphi(c)$;

  A3 $\varphi(c) \to (\exists x\ \varphi(x))$;

  A4 All generalizations of $x \approx x$;

  A5 All generalizations of $x \approx y \to (\alpha \to \alpha')$, where α is atomic and α' is obtained from α by replacing zero or more occurrences of x by y.

- *Inference rules*

  A6 Modus Ponens: $\dfrac{\psi \qquad \psi \to \theta}{\theta}$;

  A7 Generalization: $\dfrac{\psi \to \varphi(c)}{\psi \to (\forall x\ \varphi(x))}$ provided constant c does not occur in $\varphi(x)$ nor in ψ;

  A8 Generalization: $\dfrac{\varphi(c) \to \psi}{(\exists x\ \varphi(x)) \to \psi}$ provided constant c does not occur in φ nor in ψ.

Note that the axiom scheme A2 above is more restrictive than the one on page 104 of Enderton's book, which is:

$$(\forall x\ \alpha) \to \alpha^x_t \text{ provided } t \text{ is substitutable for } x \text{ in } \alpha$$

where t is not restricted to a constant symbol and, therefore, one has to worry about capture of free variables in t after substituting it for x in α. Moreover, Enderton allows α to have free variables other than x.

System B

System B is a variation on System A, but simpler, with the same notational conventions.

- *Axiom schemes*

  B1 All closed tautologies;

  B2 $(\forall x\ \varphi(x)) \to \varphi(c)$;

  B3 All generalizations of $x \approx x$;

  B4 All generalizations of $x \approx y \to (\alpha \to \alpha')$, where α is atomic and α' is obtained from α by replacing zero or more occurrences of x by y.

- *Inference rules*

  B5 Modus Ponens: $\dfrac{\psi \qquad \psi \to \theta}{\theta}$;

  B6 Generalization: $\dfrac{\varphi(c) \to \psi}{(\exists x\ \varphi(x)) \to \psi}$ provided c does not occur in φ nor in ψ .

System C

System C is a variation on System B, exhibiting a useful symmetry for some arguments, with the same notational conventions as before.

- *Axiom schemes*

  C1 All closed tautologies;

  C2 All generalizations of $x \approx x$;

  C3 All generalizations of $x \approx y \to (\alpha \to \alpha')$, where α is atomic and α' is obtained from α by replacing zero or more occurrences of x by y.

- *Inference rules*

  C4 Modus Ponens: $\dfrac{\psi \qquad \psi \to \theta}{\theta}$;

  C5 Generalization: $\dfrac{\varphi(c) \to \psi}{(\forall x\ \varphi(x)) \to \psi}$;

  C6 Generalization: $\dfrac{\varphi(c) \to \psi}{(\exists x\ \varphi(x)) \to \psi}$ provided c does not occur in φ nor in ψ .

System D

System D is a variation on System C, with the same notational conventions. But note that System D does *not* include Modus Ponens as one of its inference rules.

- *Axiom schemes*

  D1 All closed tautologies;

  D2 All generalizations of $x \approx x$;

  D3 All generalizations of $x \approx y \to (\alpha \to \alpha')$, where α is atomic and α' is obtained from α by replacing zero or more occurrences of x by y.

- *Inference rules*

  D4 Generalization: $\dfrac{((\forall x\ \varphi(x)) \to \varphi(c)) \to \psi}{\psi}$;

  D5 Generalization: $\dfrac{((\exists x\ \varphi(x)) \to \varphi(c)) \to \psi}{\psi}$ provided c does not occur in φ nor in ψ.

Reference: Without the axiom schemes for the equality symbol $\approx$, Systems A, B, C, and D are basically the systems given in Chapter VIII of [6]. In [6], all the logical connectives $\{\neg, \vee, \wedge, \to\}$ and both quantifiers $\{\exists, \forall\}$ are included as primitive symbols in the syntax of wff's, in contrast to Enderton who includes only $\{\neg, \to, \forall\}$ as primitives, leaving the other symbols $\{\vee, \wedge, \exists\}$ as abbreviations.

Exercise 1: Show that (i) the proof system in Enderton's book, in Section 2.4, is equivalent to System A, (ii) System A is equivalent to System B, (iii) System B is equivalent to System C, and (iv) System C is equivalent to System D. Thus, all of these 5 formal proof systems are equivalent, i.e. everything deducible in one is deducible in the other and vice-versa.

System E

System E is again very close to the system in Enderton's book. φ and ψ are arbitrary wff's, x and y arbitrary variables, and t an arbitrary term.

- *Axiom schemes*

  E1 All tautologies;

  E2 $(\forall x\ (\varphi \to \psi)) \to (\varphi \to (\forall x\ \psi))$ provided variable x is not free in φ ;

  E3 $(\forall x\ \varphi) \to \varphi^x_t$ provided t is substitutable for x in φ ;

  E4 $x \approx x$;

  E5 $x \approx y \to (t \approx t')$, where t' is obtained from t by replacing one occurrence of x by y ;

  E6 $x \approx y \to (\varphi \to \varphi')$, where φ is atomic and φ' is obtained from φ by replacing one occurrence of x by y ;

- *Inference rules*

  E7 Modus Ponens: $\dfrac{\varphi \qquad \varphi \to \psi}{\psi}$;

  E8 Generalization: $\dfrac{\varphi}{(\forall x\ \varphi)}$.

Reference: System E is the system in Chapter 1 of [2]. The primitive symbols in [2] are $\{\neg, \wedge, \forall\}$, so that $(\varphi \to \psi)$ (in the axiom schemes and inference rules) is the usual abbreviation for $\neg(\varphi \wedge \neg\psi)$.

Exercise 2: Show the proof system in Enderton's book, in Section 2.4, is equivalent to System E.

System F

By comparison to the preceding systems, System F has fewer axiom schemes but more inference rules. Its single propositional axiom scheme and its many inference rules make it very different from the preceding Hilbert systems, and in fact quite close to a Gentzen system (more on this in Handout 3).

- *Axiom schemes*

  F1 $\neg\alpha \vee \alpha$;

  F2 $\alpha^x_t \to (\exists x\ \alpha)$, provided t is substitutable for x in α;

  F3 $x \approx x$

  F4 $(x_1 \approx y_1 \to (\cdots \to (x_n \approx y_n \to fx_1 \cdots x_n \approx fy_1 \cdots y_n) \cdots))$, where f is any function symbol of arity $n \geq 1$;

  F5 $(x_1 \approx y_1 \to (\cdots \to (x_n \approx y_n \to (Px_1 \cdots x_n \to Py_1 \cdots y_n)) \cdots))$, where P is any predicate symbol of arity $n \geq 1$.

- *Inference rules*

  F6 Expansion: $\dfrac{\alpha}{\beta \vee \alpha}$;

  F7 Contraction: $\dfrac{\alpha \vee \alpha}{\alpha}$;

  F8 Associativity: $\dfrac{\alpha \vee (\beta \vee \gamma)}{(\alpha \vee \beta) \vee \gamma}$;

  F9 Cut Rule: $\dfrac{\alpha \vee \beta \qquad \neg\alpha \vee \gamma}{\beta \vee \gamma}$;

  F10 $\exists$-Introduction Rule: $\dfrac{\alpha \to \beta}{(\exists x\ \alpha) \to \beta}$ provided x does not occur free in β.

Reference: System F is basically the system in Chapter 2 of [5]. The primitive symbols in [5] are $\{\neg, \vee, \exists\}$, so that $(\alpha \to \beta)$ is the usual abbreviation for $(\neg\alpha \vee \beta)$.

Exercise 3: Show the proof system in Enderton's book, in Section 2.4, is equivalent to System F.

References

[1] Bell, J., and Machover, M., *A Course in Mathematical Logic*, North-Holland, Ams., 1977.

[2] Chang, C.C. and Keisler, H.J., *Model Theory*, North-Holland, Amsterdam, 1973.

[3] Kleene, S.C., *Introduction to Metamathematics*, Van Nostrand, 1952.

[4] Nerode, A and Shore, R.A., *Logic for Applications*, Springer-Verlag, New York, 1993.

[5] Shoenfield, J.R., *Mathematical Logic*, Addision-Wesley, 1967.

[6] Smullyan, R.M., *First-Order Logic*, Springer-Verlag, New York, 1968.