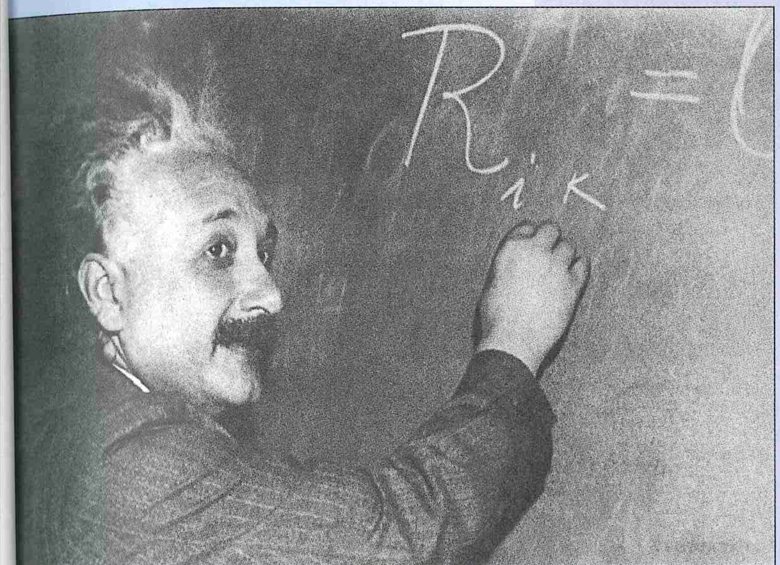
P Preliminary Concepts



Albert Einstein proposed relativity theory over 100 years ago, in 1905.

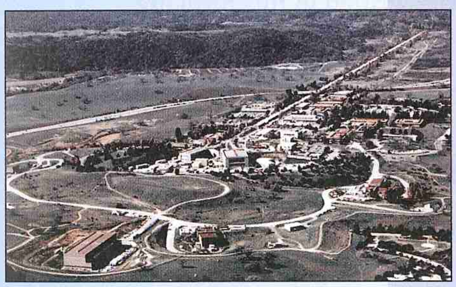
Relativity at 100 Years Old

PET (positron emission tomography) scans, the temperature of Earth's crust, smoke detectors, neon signs, carbon dating, and the warmth we receive from the sun may seem to be disparate concepts. However, they have a common theme: Albert Einstein's Theory of Special Relativity.

When Einstein was asked about his innate curiosity, he replied:

> "The important thing is not to stop questioning. Curiosity has its own reason for existing. One cannot help but be in awe when he contemplates the mysteries of eternity, of life, of the marvelous structure of reality. It is enough if one tries merely to comprehend a little of this mystery every day."

Today, relativity theory is used in conjunction with other concepts of physics to study ideas ranging from the structure of an atom to the structure of the universe. Some of Einstein's equations require working with radical expressions, such as those given in **Exercise 1 in the Project on page 33;** other equations use rational expressions, such as the expression given in **Exercise 64 on page 63.**



Stanford Linear Accelerator Center (SLAC). Atomic particles are accelerated to high speeds inside the long structure in the photo above. At speeds that approach the speed of light, physicists can confirm some of the tenets of relativity theory.

SSG

Online Study Center
For online student resources, such as section quizzes, visit this website: www.cengage.com/mathematics/aufmann

Section P.1

- Sets
- Union and Intersection of Sets
- Absolute Value and Distance
- Interval Notation
- Order of Operations Agreement
- Simplifying Variable Expressions

The Real Number System

Sets

Human beings share the desire to organize and classify. Ancient astronomers classified stars into groups called *constellations*. Modern astronomers continue to classify stars by such characteristics as color, mass, size, temperature, and distance from Earth. In mathematics it is useful to place numbers with similar characteristics into **sets**. The following sets of numbers are used extensively in the study of algebra:

Integers	$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers	{all terminating or repeating decimals}
Irrational numbers	{all nonterminating, nonrepeating decimals}
Real numbers	{all rational or irrational numbers}

If a number in decimal form terminates or repeats a block of digits, then the number is a rational number. Here are two examples of rational numbers.

0.75 is a terminating decimal.

$0.2\overline{45}$ is a repeating decimal. The bar over the 45 means that the digits 45 repeat without end. That is, $0.2\overline{45} = 0.24545454\ldots$.

Rational numbers also can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Examples of rational numbers written in this form are

$$\frac{3}{4} \qquad \frac{27}{110} \qquad -\frac{5}{2} \qquad \frac{7}{1} \qquad \frac{-4}{3}$$

Note that $\frac{7}{1} = 7$, and, in general, $\frac{n}{1} = n$ for any integer n. Therefore, all integers are rational numbers.

When a rational number is written in the form $\frac{p}{q}$, the decimal form of the rational number can be found by dividing the numerator by the denominator.

$$\frac{3}{4} = 0.75 \qquad \frac{27}{110} = 0.2\overline{45}$$

In its decimal form, an irrational number neither terminates nor repeats. For example, 0.272272227... is a nonterminating, nonrepeating decimal and thus is an irrational number. One of the best-known irrational numbers is pi, denoted by the Greek symbol π. The number π is defined as the ratio of the circumference of a circle to its diameter. Often in applications the rational number 3.14 or the rational number $\frac{22}{7}$ is used as an approximation of the irrational number π.

Every real number is either a rational number or an irrational number. If a real number is written in decimal form, it is a terminating decimal, a repeating decimal, or a nonterminating and nonrepeating decimal.

Math Matters

Archimedes (c. 287–212 B.C.) was the first to calculate π with any degree of precision. He was able to show that

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

from which we get the approximation

$$3\frac{1}{7} = \frac{22}{7} \approx \pi$$

The use of the symbol π for this quantity was introduced by Leonhard Euler (1707–1783) in 1739, approximately 2000 years after Archimedes.

The relationships among the various sets of numbers are shown in **Figure P.1**.

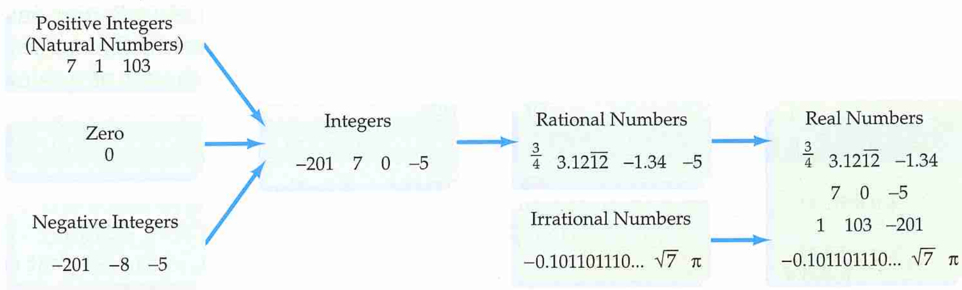


Figure P.1

Math Matters

Sophie Germain (1776–1831) was born in Paris, France. Because enrollment in the university she wanted to attend was available only to men, Germain attended under the name of Antoine-August Le Blanc. Eventually her ruse was discovered, but not before she came to the attention of Pierre Lagrange, one of the best mathematicians of the time. He encouraged her work and became a mentor to her. A certain type of prime number is named after her, called a *Germain prime number*. It is a number p such that p and $2p + 1$ are both prime. For instance, 11 is a Germain prime because $2(11) + 1 = 23$ and 11 and 23 are both prime numbers. Germain primes are used in public key cryptography, a method used to send secure communications over the Internet.

Prime numbers and *composite numbers* play an important role in almost every branch of mathematics. A **prime number** is a positive integer greater than 1 that has no positive-integer factors[1] other than itself and 1. The 10 smallest prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. Each of these numbers has only itself and 1 as factors.

A **composite number** is a positive integer greater than 1 that is not a prime number. For example, 10 is a composite number because 10 has both 2 and 5 as factors. The 10 smallest composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18.

Alternative to Example 1

For each number, check all that apply.
N = Natural
I = Integer
Q = Rational
R = Real

	N	I	Q	R
−57		✔	✔	✔
3.3719			✔	✔
7.42$\overline{917}$			✔	✔
0		✔	✔	✔
1.191191119...				✔
101	✔	✔	✔	✔

EXAMPLE 1 Classify Real Numbers

Determine which of the following numbers are

a. integers **b.** rational numbers **c.** irrational numbers
d. real numbers **e.** prime numbers **f.** composite numbers

$$-0.2, \quad 0, \quad 0.\overline{3}, \quad \pi, \quad 6, \quad 7, \quad 41, \quad 51, \quad 0.71771777177771\ldots$$

Solution

a. Integers: 0, 6, 7, 41, 51

b. Rational numbers: $-0.2, 0, 0.\overline{3}, 6, 7, 41, 51$

c. Irrational numbers: $0.71771777177771\ldots, \pi$

d. Real numbers: $-0.2, 0, 0.\overline{3}, \pi, 6, 7, 41, 51, 0.71771777177771\ldots$

e. Prime numbers: 7, 41

f. Composite numbers: 6, 51

Try Exercise 2, page 14

[1] Recall that a factor of a number divides the number evenly. For instance, 3 and 7 are factors of 21; 5 is not a factor of 21.

Each member of a set is called an **element** of the set. For instance, if $C = \{2, 3, 5\}$, then the elements of C are 2, 3, and 5. The notation $2 \in C$ is read "2 is an element of C." Set A is a **subset** of set B if every element of A is also an element of B, and we write $A \subseteq B$. For instance, the set of **negative integers** $\{-1, -2, -3, -4, \ldots\}$ is a subset of the set of integers. The set of **positive integers** $\{1, 2, 3, 4, \ldots\}$ (also known as the set of **natural numbers**) is also a subset of the set of integers.

QUESTION Are the integers a subset of the rational numbers?

take note

The order of the elements of a set is not important. For instance, the set of natural numbers less than 6 given at the right could have been written $\{3, 5, 2, 1, 4\}$. It is customary, however, to list elements of a set in numerical order.

The **empty set,** or **null set,** is the set that contains no elements. The symbol $\varnothing$ is used to represent the empty set. The set of people who have run a two-minute mile is the empty set.

The set of natural numbers less than 6 is $\{1, 2, 3, 4, 5\}$. This is an example of a **finite set;** all the elements of the set can be listed. The set of all natural numbers is an example of an **infinite set.** There is no largest natural number, so all the elements of the set of natural numbers cannot be listed.

Sets are often written using **set-builder notation.** Set-builder notation can be used to describe almost any set, but it is especially useful when writing infinite sets. For instance, the set

$$\{2n \mid n \in \text{natural numbers}\}$$

is read as "the set of elements $2n$ such that n is a natural number." By replacing n by each of the natural numbers, this is the set of positive even integers: $\{2, 4, 6, 8, \ldots\}$.

The set of real numbers greater than 2 is written

$$\{x \mid x > 2, x \in \text{real numbers}\}$$

and is read "the set of x such that x is greater than 2 and x is an element of the real numbers."

Much of the work we do in this text uses the real numbers. With this in mind, we will frequently write, for instance, $\{x \mid x > 2, x \in \text{real numbers}\}$ in a shortened form as $\{x \mid x > 2\}$, where we assume that x is a real number.

Math Matters

A **fuzzy set** is one in which each element is given a "degree" of membership. The concepts behind fuzzy sets are used in a wide variety of applications such as traffic lights, washing machines, and computer speech recognition programs.

Alternative to Example 2

List the four smallest elements in $\{n^2 \mid n \in \text{integers}\}$.

■ **0, 1, 4, 9**

EXAMPLE 2 ⟫ Use Set-Builder Notation

List the four smallest elements in $\{n^3 \mid n \in \text{natural numbers}\}$.

Solution

Because we want the four *smallest* elements, we choose the four smallest natural numbers. Thus $n = 1, 2, 3$, and 4. Therefore, the four smallest elements of $\{n^3 \mid n \in \text{natural numbers}\}$ are 1, 8, 27, and 64.

⟫ *Try Exercise 6, page 14*

ANSWER Yes.

Union and Intersection of Sets

Just as operations such as addition and multiplication are performed on real numbers, operations are performed on sets. Two operations performed on sets are union and intersection. The union of two sets A and B is the set of elements that belong to A or to B, or to both A and B.

Definition of the Union of Two Sets

The **union** of two sets, written $A \cup B$, is the set of all elements that belong to either A or B. In set-builder notation, this is written

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Example

Given $A = \{2, 3, 4, 5\}$ and $B = \{0, 1, 2, 3, 4\}$, find $A \cup B$.

$A \cup B = \{0, 1, 2, 3, 4, 5\}$ • Note that an element that belongs to both sets is listed only once.

The intersection of the two sets A and B is the set of elements that belong to both A and B.

Definition of the Intersection of Two Sets

The **intersection** of two sets, written $A \cap B$, is the set of all elements that are common to both A and B. In set-builder notation, this is written

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Example

Given $A = \{2, 3, 4, 5\}$ and $B = \{0, 1, 2, 3, 4\}$, find $A \cap B$.

$A \cap B = \{2, 3, 4\}$ • The intersection of two sets contains the elements common to both sets.

If the intersection of two sets is the empty set, the two sets are said to be **disjoint**. For example, if $A = \{2, 3, 4\}$ and $B = \{7, 8\}$, then $A \cap B = \varnothing$ and A and B are disjoint sets.

Alternative to Example 3
Given $A = \{-3, 0, 3, 6, 9\}$, $B = \{-3, -2, -1, 0, 1, 2, 3\}$, and $C = \{5, 6, 7\}$, find

a. $A \cup B$

■ $\{-3, -2, -1, 0, 1, 2, 3, 6, 9\}$

b. $B \cap (A \cup C)$

■ $\{-3, 0, 3\}$

c. $B \cap C$

■ $\varnothing$

EXAMPLE 3 Find the Union and Intersection of Sets

Find each intersection or union given $A = \{0, 2, 4, 6, 10, 12\}$, $B = \{0, 3, 6, 12, 15\}$, and $C = \{1, 2, 3, 4, 5, 6, 7\}$.

a. $A \cup C$ **b.** $B \cap C$ **c.** $A \cap (B \cup C)$ **d.** $B \cup (A \cap C)$

Solution

a. $A \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 10, 12\}$ • The elements that belong to A or C

b. $B \cap C = \{3, 6\}$ • The elements that belong to B and C

Continued ▶

c. First determine $B \cup C = \{0, 1, 2, 3, 4, 5, 6, 7, 12, 15\}$. Then

$A \cap (B \cup C) = \{0, 2, 4, 6, 12\}$ • **The elements that belong to A and $(B \cup C)$**

d. First determine $A \cap C = \{2, 4, 6\}$. Then

$B \cup (A \cap C) = \{0, 2, 3, 4, 6, 12, 15\}$ • **The elements that belong to B or $(A \cap C)$**

Try Exercise 16, page 14

Absolute Value and Distance

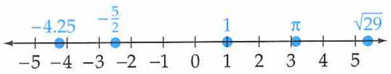


Figure P.2

The real numbers can be represented geometrically by a **coordinate axis** called a **real number line. Figure P.2** shows a portion of a real number line. The number associated with a point on a real number line is called the **coordinate** of the point. The point corresponding to zero is called the **origin.** Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number.

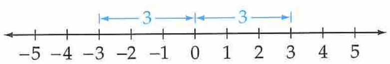


Figure P.3

The *absolute value* of a real number a, denoted $|a|$, is the distance between a and 0 on the number line. For instance, $|3| = 3$ and $|-3| = 3$ because both 3 and -3 are 3 units from zero. See **Figure P.3.**

In general, if $a \geq 0$, then $|a| = a$; however, if $a < 0$, then $|a| = -a$ because $-a$ is positive when $a < 0$. This leads to the following definition.

take note

The second part of the definition of absolute value states that if $a < 0$, then $|a| = -a$. For instance, if $a = -4$, then

$$|a| = |-4| = -(-4) = 4$$

Definition of Absolute Value

The **absolute value** of the real number a is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Example

$|5| = 5 \qquad |-4| = 4 \qquad |0| = 0$

The definition of *distance* between two points on a real number line makes use of absolute value.

Definition of the Distance Between Points on a Real Number Line

If a and b are the coordinates of two points on a real number line, the **distance** between the graph of a and the graph of b, denoted by $d(a, b)$, is given by $d(a, b) = |a - b|$.

Example

Find the distance between a point whose coordinate on the real number line is -2 and a point whose coordinate is 5.

$$d(-2, 5) = |-2 - 5| = |-7| = 7$$

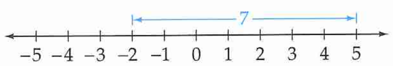


Figure P.4

Note in **Figure P.4** that there are 7 units between -2 and 5. Also note that the order of the coordinates in the formula does not matter.

$$d(5, -2) = |5 - (-2)| = |7| = 7$$

Alternative to Example 4

Express "the distance between x and 4 is less than 5" using absolute value notation.

■ $|x - 4| < 5$

EXAMPLE 4 » **Use Absolute Value to Express the Distance Between Two Points**

Express the distance between a and -3 on the number line using absolute value.

Solution

$$d(a, -3) = |a - (-3)| = |a + 3|$$

» *Try Exercise 48, page 15*

Interval Notation

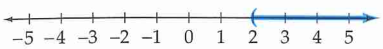


Figure P.5

The graph of $\{x \mid x > 2\}$ is shown in **Figure P.5.** The set is the real numbers greater than 2. The parenthesis at 2 indicates that 2 is not included in the set. Rather than write this set of real numbers using set-builder notation, we can write the set in **interval notation** as $(2, \infty)$.

In general, the interval notation

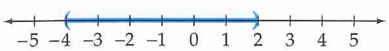


Figure P.6

(a, b) represents all real numbers between a and b, not including a and b. This is an **open interval.** In set-builder notation, we write $\{x \mid a < x < b\}$. The graph of $(-4, 2)$ is shown in **Figure P.6.**

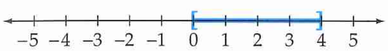


Figure P.7

$[a, b]$ represents all real numbers between a and b, including a and b. This is a **closed interval.** In set-builder notation, we write $\{x \mid a \le x \le b\}$. The graph of $[0, 4]$ is shown in **Figure P.7.** The brackets at 0 and 4 indicate that those numbers are included in the graph.

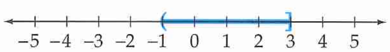


Figure P.8

$(a, b]$ represents all real numbers between a and b, not including a but including b. This is a **half-open interval.** In set-builder notation, we write $\{x \mid a < x \le b\}$. The graph of $(-1, 3]$ is shown in **Figure P.8.**

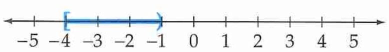


Figure P.9

$[a, b)$ represents all real numbers between a and b, including a but not including b. This is a **half-open interval.** In set-builder notation, we write $\{x \mid a \le x < b\}$. The graph of $[-4, -1)$ is shown in **Figure P.9.**

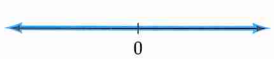
Subsets of the real numbers whose graphs extend forever in one or both directions can be represented by interval notation using the **infinity symbol** ∞ or the **negative infinity symbol** $-\infty$.

$(-\infty, a)$ represents all real numbers less than a.

(b, ∞) represents all real numbers greater than b.

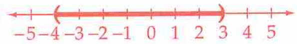
$(-\infty, a]$ represents all real numbers less than or equal to a.

$[b, \infty)$ represents all real numbers greater than or equal to b.



$(-\infty, \infty)$ represents all real numbers.

Alternative to Example 5
Graph the interval $(-4, 3)$. Write the interval in set-builder notation.

■ 

$\{x \mid -4 < x < 3\}$

take note

It is *never* correct to use a bracket when using the infinity symbol. For instance, $[-\infty, 3]$ is not correct. Nor is $[2, \infty]$ correct. Neither negative infinity nor positive infinity is a real number and therefore cannot be contained in an interval.

EXAMPLE 5 ›› Graph a Set Given in Interval Notation

Graph $(-\infty, 3]$. Write the interval in set-builder notation.

Solution

The set is the real numbers less than or equal to 3. In set-builder notation, this is the set $\{x \mid x \leq 3\}$. Draw a right bracket at 3, and darken the number line to the left of 3, as shown in **Figure P.10.**

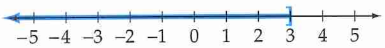


Figure P.10

›› *Try Exercise 54, page 15*

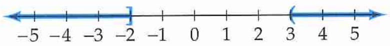


Figure P.11

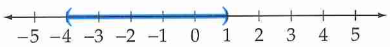


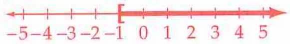
Figure P.12

The set $\{x \mid x \leq -2\} \cup \{x \mid x > 3\}$ is the set of real numbers that are either less than or equal to -2 or greater than 3. We also could write this in interval notation as $(-\infty, -2] \cup (3, \infty)$. The graph of the set is shown in **Figure P.11.**

The set $\{x \mid x > -4\} \cap \{x \mid x < 1\}$ is the set of real numbers that are greater than -4 *and* less than 1. Note from **Figure P.12** that this set is the interval $(-4, 1)$, which can be written in set-builder notation as $\{x \mid -4 < x < 1\}$.

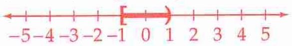
Alternative to Example 6
Graph the following. Write **a.** using interval notation and write **b.** using set-builder notation.

a. $\{x \mid x \geq -1\} \cup \{x \mid x \geq 3\}$

■ 

$[-1, \infty)$

b. $(-\infty, 1) \cap [-1, 3]$

■ 

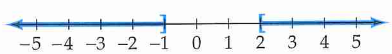
$\{x \mid -1 \leq x < 1\}$

EXAMPLE 6 ›› Graph Intervals

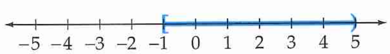
Graph the following. Write **a.** and **b.** using interval notation. Write **c.** and **d.** using set-builder notation.

a. $\{x \mid x \leq -1\} \cup \{x \mid x \geq 2\}$ **b.** $\{x \mid x \geq -1\} \cap \{x \mid x < 5\}$
c. $(-\infty, 0) \cup [1, 3]$ **d.** $[-1, 3] \cap (1, 5)$

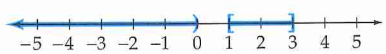
Solution

a. 

$(-\infty, -1] \cup [2, \infty)$

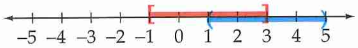
b. 

$[-1, 5)$

c. 

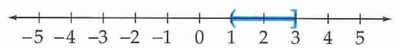
$\{x \mid x < 0\} \cup \{x \mid 1 \leq x \leq 3\}$

d. The graphs of $[-1, 3]$, in red, and $(1, 5)$, in blue, are shown below.



Note that the intersection of the sets occurs where the graphs intersect. Although $1 \in [-1, 3]$, $1 \notin (1, 5)$. Therefore, 1 does not belong to the intersection of the sets. On the other hand, $3 \in [-1, 3]$ and $3 \in (1, 5)$.

Therefore, 3 belongs to the intersection of the sets. Thus we have the following.



$\{x \mid 1 < x \leq 3\}$

» Try Exercise 64, page 15

Order of Operations Agreement

The approximate pressure p, in pounds per square inch, on a scuba diver x feet below the water's surface is given by

$$p = 15 + 0.5x$$

The pressure on the diver at various depths is given below.

10 feet	$15 + 0.5(10) = 15 + 5 = 20$ pounds
20 feet	$15 + 0.5(20) = 15 + 10 = 25$ pounds
40 feet	$15 + 0.5(40) = 15 + 20 = 35$ pounds
70 feet	$15 + 0.5(70) = 15 + 35 = 50$ pounds

Note that the expression $15 + 0.5(70)$ has two operations, addition and multiplication. When an expression contains more than one operation, the operations must be performed in a specified order, as given by the Order of Operations Agreement.

The Order of Operations Agreement

If grouping symbols are present, evaluate by first performing the operations within the grouping symbols, innermost grouping symbols first, while observing the order given in steps 1 to 3.

Step 1 Evaluate exponential expressions.
Step 2 Do multiplication and division as they occur from left to right.
Step 3 Do addition and subtraction as they occur from left to right.

Example

Evaluate: $5 - 7(23 - 5^2) - 16 \div 2^3$

$5 - 7(23 - 5^2) - 16 \div 2^3$

$= 5 - 7(23 - 25) - 16 \div 2^3$ • **Begin inside the parentheses and evaluate $5^2 = 25$.**

$= 5 - 7(-2) - 16 \div 2^3$ • **Continue inside the parentheses and evaluate $23 - 25 = -2$.**

$= 5 - 7(-2) - 16 \div 8$ • **Evaluate $2^3 = 8$.**

$= 5 - (-14) - 2$ • **Perform multiplication and division from left to right.**

$= 17$ • **Perform addition and subtraction from left to right.**

> **take note**
>
> Recall that subtraction can be rewritten as addition of the opposite. Therefore,
>
> $3x^2 - 4xy + 5x - y - 7$
> $= 3x^2 + (-4xy) + 5x + (-y) + (-7)$
>
> In this form, we can see that the terms (addends) are $3x^2$, $-4xy$, $5x$, $-y$, and -7.

One of the ways in which the Order of Operations Agreement is used is to evaluate variable expressions. The addends of a variable expression are called **terms.** The terms for the expression at the right are $3x^2$, $-4xy$, $5x$, $-y$, and -7. Observe that the sign of a term is the sign that immediately precedes it.

$$3x^2 - 4xy + 5x - y - 7$$

The terms $3x^2$, $-4xy$, $5x$, and $-y$ are **variable terms.** The term -7 is a **constant term.** Each variable term has a **numerical coefficient** and a **variable part.** The numerical coefficient for the term $3x^2$ is 3; the numerical coefficient for the term $-4xy$ is -4; the numerical coefficient for the term $5x$ is 5; and the numerical coefficient for the term $-y$ is -1. When the numerical coefficient is 1 or -1 (as in x and $-x$), the 1 is usually not written.

To **evaluate** a variable expression, replace the variables by their given values and then use the Order of Operations Agreement to simplify the result.

Alternative to Example 7
Evaluate $3ab - 4(2a - 3b)$ when $a = 4$ and $b = -3$.
■ **−104**

EXAMPLE 7 Evaluate a Variable Expression

a. Evaluate $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$ when $x = 2$ and $y = -3$.

b. Evaluate $(x + 2y)^2 - 4z$ when $x = 3$, $y = -2$, and $z = -4$.

Solution

a. $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$

$$\frac{2^3 - (-3)^3}{2^2 + 2(-3) + (-3)^2} = \frac{8 - (-27)}{4 - 6 + 9} = \frac{35}{7} = 5$$

b. $(x + 2y)^2 - 4z$

$$(3 + 2(-2))^2 - 4(-4) = (3 + (-4))^2 - 4(-4)$$
$$= (-1)^2 - 4(-4) = 1 - 4(-4)$$
$$= 1 + 16 = 17$$

Try Exercise 74, page 15

Simplifying Variable Expressions

Addition, multiplication, subtraction, and *division* are the operations of arithmetic. **Addition** of the two real numbers a and b is designated by $a + b$. If $a + b = c$, then c is the **sum** and the real numbers a and b are called **terms.**

Multiplication of the real numbers a and b is designated by ab or $a \cdot b$. If $ab = c$, then c is the **product** and the real numbers a and b are called **factors** of c.

The number $-b$ is referred to as the **additive inverse** of b. **Subtraction** of the real numbers a and b is designated by $a - b$ and is defined as the sum of a and the additive inverse of b. That is,

$$a - b = a + (-b)$$

If $a - b = c$, then c is called the **difference** of a and b.

The **multiplicative inverse** or **reciprocal** of the nonzero number b is $1/b$. The **division** of a and b, designated by $a \div b$ with $b \neq 0$, is defined as the product of a and the reciprocal of b. That is,

$$a \div b = a\left(\frac{1}{b}\right) \qquad \text{provided that } b \neq 0$$

If $a \div b = c$, then c is called the **quotient** of a and b.

The notation $a \div b$ is often represented by the fractional notation a/b or $\dfrac{a}{b}$. The real number a is the **numerator,** and the nonzero real number b is the **denominator** of the fraction.

Properties of Real Numbers

Let a, b, and c be real numbers.

	Addition Properties	Multiplication Properties
Closure	$a + b$ is a unique real number.	ab is a unique real number.
Commutative	$a + b = b + a$	$ab = ba$
Associative	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
Identity	There exists a unique real number 0 such that $a + 0 = 0 + a = a$.	There exists a unique real number 1 such that $a \cdot 1 = 1 \cdot a = a$.
Inverse	For each real number a, there is a unique real number $-a$ such that $a + (-a) = (-a) + a = 0$.	For each *nonzero* real number a, there is a unique real number $1/a$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.
Distributive	$a(b + c) = ab + ac$	

Alternative to Example 8

Identify the property.

a. $a(bc) = a(cb)$

■ **Commutative property of multiplication**

b. $a + (-a) = 0$

■ **Inverse property of addition**

c. $c + 0 = c$

■ **Identity property of addition**

d. $3(x + y) = 3(y + x)$

■ **Commutative property of addition**

EXAMPLE 8 Identify Properties of Real Numbers

Identify the property of real numbers illustrated in each statement.

a. $(2a)b = 2(ab)$

b. $\left(\frac{1}{5}\right)11$ is a real number.

c. $4(x + 3) = 4x + 12$

d. $(a + 5b) + 7c = (5b + a) + 7c$

e. $\left(\frac{1}{2} \cdot 2\right)a = 1 \cdot a$

f. $1 \cdot a = a$

Continued ▶

Solution

a. Associative property of multiplication

b. Closure property of multiplication of real numbers

c. Distributive property

d. Commutative property of addition

e. Inverse property of multiplication

f. Identity property of multiplication

Try Exercise 86, page 15

We can identify which properties of real numbers have been used to rewrite an expression by closely comparing the original and final expressions and noting any changes. For instance, to simplify $(6x)2$, both the commutative property and associative property of multiplication are used.

$$(6x)2 = 2(6x) \quad \text{• Commutative property of multiplication}$$
$$= (2 \cdot 6)x \quad \text{• Associative property of multiplication}$$
$$= 12x$$

To simplify $3(4p + 5)$, use the distributive property.

$$3(4p + 5) = 3(4p) + 3(5) \quad \text{• Distributive property}$$
$$= 12p + 15$$

Terms that have the same variable part are called **like terms.** The distributive property is also used to simplify an expression with like terms such as $3x^2 + 9x^2$.

$$3x^2 + 9x^2 = (3 + 9)x^2 \quad \text{• Distributive property}$$
$$= 12x^2$$

Note from this example that like terms are combined by adding the coefficients of the like terms.

> **take note**
>
> Normally, we will not show, as we did at the right, all the steps involved in the simplification of a variable expression. For instance, we will just write $(6x)2 = 12x$, $3(4p + 5) = 12p + 15$, and $3x^2 + 9x^2 = 12x^2$. It is important to know, however, that every step in the simplification process depends on one of the properties of real numbers.

QUESTION Are the terms $2x^2$ and $3x$ like terms?

EXAMPLE 9 Simplify Variable Expressions

a. Simplify $5 + 3(2x - 6)$.

b. Simplify $4x - 2[7 - 5(2x - 3)]$.

Alternative to Example 9

Simplify the following.

a. $6 - 3(5a - 4)$

■ $-15a + 18$

b. $-3(2a - 5b + 1) + 5(3a - b + 2)$

■ $9a + 10b + 7$

ANSWER No. The variable parts are not the same. The variable part of $2x^2$ is $x \cdot x$. The variable part of $3x$ is x.

Solution

a. $5 + 3(2x - 6) = 5 + 6x - 18$ • Use the distributive property.

$= 6x - 13$ • Add the constant terms.

b. $4x - 2[7 - 5(2x - 3)]$

$= 4x - 2[7 - 10x + 15]$ • Use the distributive property to remove the inner parentheses.

$= 4x - 2[-10x + 22]$ • Simplify.

$= 4x + 20x - 44$ • Use the distributive property to remove the brackets.

$= 24x - 44$ • Simplify.

Try Exercise 106, page 16

An **equation** is a statement of equality between two numbers or two expressions. There are four basic properties of equality that relate to equations.

Properties of Equality

Let a, b, and c be real numbers.

Reflexive	$a = a$
Symmetric	If $a = b$, then $b = a$.
Transitive	If $a = b$ and $b = c$, then $a = c$.
Substitution	If $a = b$, then a may be replaced by b in any expression that involves a.

Alternative to Example 10
Identify the property of equality illustrated in the following statement:
If $2x + 3y = 7$ and $7 = x^2 + y^2$, then $2x + 3y = x^2 + y^2$.
■ **Transitive**

EXAMPLE 10 Identify Properties of Equality

Identify the property of equality illustrated in each statement.

a. If $3a + b = c$, then $c = 3a + b$.

b. $5(x + y) = 5(x + y)$

c. If $4a - 1 = 7b$ and $7b = 5c + 2$, then $4a - 1 = 5c + 2$.

d. If $a = 5$ and $b(a + c) = 72$, then $b(5 + c) = 72$.

Solution

a. Symmetric **b.** Reflexive **c.** Transitive **d.** Substitution

Try Exercise 90, page 15

Topics for Discussion

1. Archimedes determined that $\frac{223}{71} < \pi < \frac{22}{7}$. Is it possible to find an exact representation for π of the form $\frac{a}{b}$, where a and b are integers?
2. If $I = \{\text{irrational numbers}\}$ and $Q = \{\text{rational numbers}\}$, name the sets $I \cup Q$ and $I \cap Q$.
3. If the proposed simplification shown at the right is correct, so state. If it is incorrect, show a correct simplification. $2 \cdot 3^2 = 6^2 = 36$
4. Are there any even prime numbers? If so, name them.
5. Does every real number have an additive inverse? Does every real number have a multiplicative inverse?
6. What is the difference between an open interval and a closed interval?

—Suggested Assignment: Exercises 1–105, every other odd; 107–123, odd; 127, 129.
—Answers for Exercises 1, 2, 19–30, and 51–66 are on page AA1.

Exercise Set P.1

In Exercises 1 and 2, determine whether each number is an integer, a rational number, an irrational number, a prime number, or a real number.

1. $-\frac{1}{5}, 0, -44, \pi, 3.14, 5.05005000500005\ldots, \sqrt{81}, 53$
2. $\frac{5}{\sqrt{7}}, \frac{5}{7}, 31, -2\frac{1}{2}, 4.235653907493, 51, 0.888\ldots$

In Exercises 3 to 8, list the four smallest elements of each set.

3. $\{2x \mid x \in \text{positive integers}\}$ 2, 4, 6, 8
4. $\{|x| \mid x \in \text{integers}\}$ 0, 1, 2, 3
5. $\{y \mid y = 2x + 1, x \in \text{natural numbers}\}$ 3, 5, 7, 9
6. $\{y \mid y = x^2 - 1, x \in \text{integers}\}$ −1, 0, 3, 8
7. $\{z \mid z = |x|, x \in \text{integers}\}$ 0, 1, 2, 3
8. $\{z \mid z = |x| - x, x \in \text{negative integers}\}$ 2, 4, 6, 8

In Exercises 9 to 18, perform the operations given that $A = \{-3, -2, -1, 0, 1, 2, 3\}$, $B = \{-2, 0, 2, 4, 6\}$, $C = \{0, 1, 2, 3, 4, 5, 6\}$, and $D = \{-3, -1, 1, 3\}$.

9. $A \cup B$ $\{-3, -2, -1, 0, 1, 2, 3, 4, 6\}$
10. $C \cup D$ $\{-3, -1, 0, 1, 2, 3, 4, 5, 6\}$
11. $A \cap C$ $\{0, 1, 2, 3\}$
12. $C \cap D$ $\{1, 3\}$
13. $B \cap D$ $\varnothing$
14. $B \cup (A \cap C)$ $\{-2, 0, 1, 2, 3, 4, 6\}$
15. $D \cap (B \cup C)$ $\{1, 3\}$
16. $(A \cap B) \cup (A \cap C)$ $\{-2, 0, 1, 2, 3\}$
17. $(B \cup C) \cap (B \cup D)$ $\{-2, 0, 1, 2, 3, 4, 6\}$
18. $(A \cap C) \cup (B \cap D)$ $\{0, 1, 2, 3\}$

In Exercises 19 to 30, graph each set. Write sets given in interval notation in set-builder notation, and write sets given in set-builder notation in interval notation.

19. $(-2, 3)$ $\{x \mid -2 < x < 3\}$
20. $[1, 5]$ $\{x \mid 1 \le x \le 5\}$
21. $[-5, -1]$ $\{x \mid -5 \le x \le -1\}$
22. $(-3, 3)$ $\{x \mid -3 < x < 3\}$
23. $[2, \infty)$ $\{x \mid x \ge 2\}$
24. $(-\infty, 4)$ $\{x \mid x < 4\}$
25. $\{x \mid 3 < x < 5\}$ $(3, 5)$
26. $\{x \mid x < -1\}$ $(-\infty, -1)$

27. $\{x \mid x \geq -2\}$ $[-2, \infty)$

28. $\{x \mid -1 \leq x < 5\}$ $[-1, 5)$

29. $\{x \mid 0 \leq x \leq 1\}$ $[0, 1]$

30. $\{x \mid -4 < x \leq 5\}$ $(-4, 5]$

In Exercises 31 to 40, write each expression without absolute value symbols.

31. $-|-5|$ -5

32. $-|-4|^2$ -16

33. $|3| \cdot |-4|$ 12

34. $|3| - |-7|$ -4

35. $|\pi^2 + 10|$ $\pi^2 + 10$

36. $|\pi^2 - 10|$ $10 - \pi^2$

37. $|x - 4| + |x + 5|$, given $0 < x < 1$ 9

38. $|x + 6| + |x - 2|$, given $2 < x < 3$ $2x + 4$

39. $|2x| - |x - 1|$, given $0 < x < 1$ $3x - 1$

40. $|x + 1| + |x - 3|$, given $x > 3$ $2x - 2$

In Exercises 41 to 50, use absolute value notation to describe the given situation.

41. The distance between x and 3 $|x - 3|$

42. The distance between a and -2 $|a + 2|$

43. The distance between x and -2 is 4. $|x + 2| = 4$

44. The distance between z and 5 is 1. $|z - 5| = 1$

45. $d(m, n)$ $|m - n|$

46. $d(p, 8)$ $|p - 8|$

47. The distance between a and 4 is less than 5. $|a - 4| < 5$

48. The distance between z and 5 is greater than 7. $|z - 5| > 7$

49. The distance between x and -2 is greater than 4. $|x + 2| > 4$

50. The distance between y and -3 is greater than 6. $|y + 3| > 6$

In Exercises 51 to 66, graph each set.

51. $(-\infty, 0) \cup [2, 4]$

52. $(-3, 1) \cup (3, 5)$

53. $(-4, 0) \cap [-2, 5]$

54. $(-\infty, 3] \cap (2, 6)$

55. $(1, \infty) \cup (-2, \infty)$

56. $(-4, \infty) \cup (0, \infty)$

57. $(1, \infty) \cap (-2, \infty)$

58. $(-4, \infty) \cap (0, \infty)$

59. $[-2, 4] \cap [4, 5]$

60. $(-\infty, 1] \cap [1, \infty)$

61. $(-2, 4) \cap (4, 5)$

62. $(-\infty, 1) \cap (1, \infty)$

63. $\{x \mid x < -3\} \cup \{x \mid 1 < x < 2\}$

64. $\{x \mid -3 \leq x < 0\} \cup \{x \mid x \geq 2\}$

65. $\{x \mid x < -3\} \cup \{x \mid x < 2\}$

66. $\{x \mid x < -3\} \cap \{x \mid x < 2\}$

In Exercises 67 to 78, evaluate the variable expression for $x = 3$, $y = -2$, and $z = -1$.

67. $-y^3$ 8

68. $-y^2$ -4

69. $2xyz$ 12

70. $-3xz$ 9

71. $-2x^2y^2$ -72

72. $2y^3z^2$ -16

73. $xy - z(x - y)^2$ 19

74. $(z - 2y)^2 - 3z^3$ 12

75. $\dfrac{x^2 + y^2}{x + y}$ 13

76. $\dfrac{2xy^2z^4}{(y - z)^4}$ 24

77. $\dfrac{3y}{x} - \dfrac{2z}{y}$ -3

78. $(x - z)^2(x + z)^2$ 64

In Exercises 79 to 92, state the property of real numbers or the property of equality that is used.

79. $(ab^2)c = a(b^2c)$ Associative property of multiplication

80. $2x - 3y = -3y + 2x$ Commutative property of addition

81. $4(2a - b) = 8a - 4b$ Distributive property

82. $6 + (7 + a) = 6 + (a + 7)$ Commutative property of addition

83. $(3x)y = y(3x)$ Commutative property of multiplication

84. $4ab + 0 = 4ab$ Identity property of addition

85. $1 \cdot (4x) = 4x$ Identity property of multiplication

86. $7(a + b) = 7(b + a)$ Commutative property of addition

87. $x^2 + 1 = x^2 + 1$ Reflexive property of equality

88. If $a + b = 2$, then $2 = a + b$. Symmetric property of equality

89. If $2x + 1 = y$ and $y = 3x - 2$, then $2x + 1 = 3x - 2$. Transitive property of equality

90. If $4x + 2y = 7$ and $x = 3$, then $4(3) + 2y = 7$. Substitution property of equality

91. $4 \cdot \dfrac{1}{4} = 1$ Inverse property of multiplication

92. $ab + (-ab) = 0$ Inverse property of addition

In Exercises 93 to 106, simplify the variable expression.

93. $3(2x)$ $6x$

94. $-2(4y)$ $-8y$

95. $3(2 + x)$ $3x + 6$

96. $-2(4 + y)$ $-2y - 8$

97. $\frac{2}{3}a + \frac{5}{6}a$ $\frac{3}{2}a$

98. $\frac{3}{4}x - \frac{1}{2}x$ $\frac{1}{4}x$

99. $2 + 3(2x - 5)$ $6x - 13$

100. $4 + 2(2a - 3)$ $4a - 2$

101. $5 - 3(4x - 2y)$ $-12x + 6y + 5$

102. $7 - 2(5n - 8m)$ $16m - 10n + 7$

103. $3(2a - 4b) - 4(a - 3b)$ $2a$

104. $5(4r - 7t) - 2(10r + 3t)$ $-41t$

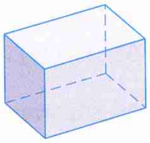
105. $5a - 2[3 - 2(4a + 3)]$ $21a + 6$

106. $6 + 3[2x - 4(3x - 2)]$ $-30x + 30$

107. **AREA OF A TRIANGLE** The area of a triangle is given by area $= \frac{1}{2}bh$, where b is the base of the triangle and h is its height. Find the area of a triangle whose base is 3 inches and whose height is 4 inches. 6 in.^{2}

108. **VOLUME OF A BOX** The volume of a rectangular box is given by

$$\text{Volume} = lwh$$



where l is the length, w is the width, and h is the height of the box. Find the volume of a classroom that has a length of 40 feet, a width of 30 feet, and a height of 12 feet. 1440 ft^3

109. **PROFIT FROM SALES** The profit, in dollars, a company earns from the sale of x bicycles is given by

$$\text{Profit} = -0.5x^2 + 120x - 2000$$

Find the profit the company earns from selling 110 bicycles. \$5150

110. **MAGAZINE CIRCULATION** The circulation, in thousands of subscriptions, of a new magazine n months after its introduction can be approximated by

$$\text{Circulation} = \sqrt{n^2 - n + 1}$$

Find, to the nearest hundred, the circulation of the magazine after 12 months. 11,500 subscriptions

111. **HEART RATE** The heart rate, in beats per minute, of a certain runner during a cool-down period can be approximated by

$$\text{Heart rate} = 65 + \frac{53}{4t + 1}$$

where t is the number of minutes after the start of cool-down. Find the runner's heart rate after 10 minutes. Round to the nearest whole number. 66 beats per minute

112. **BODY MASS INDEX** According to the National Institutes of Health, body mass index (BMI) is measure of body fat based on height and weight that applies to both adult men and women, with values between 18.5 and 24.9 considered healthy. BMI is calculated as $\text{BMI} = \frac{705w}{h^2}$, where w is the person's weight in pounds and h is the person's height in inches. Find the BMI for a person who weighs 160 pounds and is 5 feet 10 inches tall. Round to the nearest whole number. 23

113. **PHYSICS** The height, in feet, of a ball t seconds after it is thrown upward is given by height $= -16t^2 + 80t + 4$. Find the height of the ball 2 seconds after it has been released. 100 ft

114. **CHEMISTRY** Salt is being added to water in such a way that the concentration, in grams per liter, is given by concentration $= \frac{50t}{t + 1}$, where t is the time in minutes after the introduction of the salt. Find the concentration of salt after 24 minutes. 48 grams per liter

Connecting Concepts

In Exercises 115 to 118, perform the operation, given A is any set.

115. $A \cup A$ A

116. $A \cap A$ A

117. $A \cap \varnothing$ $\varnothing$

118. $A \cup \varnothing$ A

119. If A and B are two sets and $A \cup B = A$, what can be said about B? B is a subset of A.

120. If A and B are two sets and $A \cap B = B$, what can be said about B? B is a subset of A.

121. Is division of real numbers an associative operation? Give a reason for your answer.
No. $(8 \div 4) \div 2 = 2 \div 2 = 1, 8 \div (4 \div 2) = 8 \div 2 = 4$

122. Is subtraction of real numbers a commutative operation? Give a reason for your answer.
No. $5 - 3 = 2, 3 - 5 = -2$

123. Which of the properties of real numbers are satisfied by the integers? All but the multiplicative inverse property

124. Which of the properties of real numbers are satisfied by the rational numbers? All

In Exercises 125 and 126, write each expression without absolute value symbols.

125. $\left|\dfrac{x+7}{|x| + |x-1|}\right|$, given $0 < x < 1$ $x + 7$

126. $\left|\dfrac{x+3}{\left|x - \frac{1}{2}\right| + \left|x + \frac{1}{2}\right|}\right|$, given $0 < x < 0.2$ $x + 3$

In Exercises 127 to 132, use absolute value notation to describe the given statement.

127. x is closer to 2 than it is to 6. $|x - 2| < |x - 6|$

128. x is closer to a than it is to b. $|x - a| < |x - b|$

129. x is farther from 3 than it is from -7. $|x - 3| > |x + 7|$

130. x is farther from 0 than it is from 5. $|x| > |x - 5|$

131. x is more than 2 units from 4 but less than 7 units from 4.
$2 < |x - 4| < 7$

132. x is more than b units from a but less than c units from a.
$b < |x - a| < c$

Projects

1. **NUMBER PUZZLE** A number n has the following properties:

When n is divided by 6, the remainder is 5.
When n is divided by 5, the remainder is 4.
When n is divided by 4, the remainder is 3.
When n is divided by 3, the remainder is 2.
When n is divided by 2, the remainder is 1.

What is the smallest value of n?

2. **OPERATIONS ON INTERVALS** Besides finding unions and intersections of intervals, it is possible to apply other operations to intervals. For instance, $(-1, 2)^2$ is the interval that results from squaring every number in the interval $(-1, 2)$. This gives $[0, 4)$. Thus $(-1, 2)^2 = [0, 4)$.

a. Find $(-4, 2)^2$.

b. Find $ABS(-4, 5)$, the absolute value of every number in $(-4, 5)$.

c. Find $\sqrt{(0, 9)}$, the square root of every number in $(0, 9)$.

d. Find $\dfrac{1}{(0, 1)}$, the reciprocal of every number in $(0, 1)$.

3. **FACTORS OF A NUMBER** Explain why the square of a natural number always has an odd number of natural number factors.

Section P.2

- Properties of Exponents
- Scientific Notation
- Rational Exponents and Radicals
- Simplify Radical Expressions

Integer and Rational Number Exponents

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A1.

PS1. Simplify: $2^2 \cdot 2^3$ [P.1] 32

PS2. Simplify: $\frac{4^3}{4^5}$ [P.1] $\frac{1}{16}$

PS3. Simplify: $(2^3)^2$ [P.1] 64

PS4. Simplify: $3.14(10^5)$ [P.1] 314,000

PS5. True or false: $3^4 \cdot 3^2 = 9^6$ [P.1] False

PS6. True or false: $(3 + 4)^2 = 3^2 + 4^2$ [P.1] False

Properties of Exponents

A compact method of writing $5 \cdot 5 \cdot 5 \cdot 5$ is 5^4. The expression 5^4 is written in **exponential notation.** Similarly, we can write

$$\frac{2x}{3} \cdot \frac{2x}{3} \cdot \frac{2x}{3} \quad \text{as} \quad \left(\frac{2x}{3}\right)^3$$

Exponential notation can be used to express the product of any expression that is used repeatedly as a factor.

Math Matters

The expression 10^{100} is called a *googol*. The term was coined by the 9-year-old nephew of the American mathematician Edward Kasner. Many calculators do not provide for numbers of this magnitude, but it is no serious loss. To appreciate the magnitude of a googol, consider that if all the atoms in the known universe were counted, the number would not even be close to a googol. But if a googol is too small for you, try 10^{googol}, which is called a *googolplex*. As a final note, the name of the Internet site Google.com is a takeoff on the word googol.

Definition of Natural Number Exponents

If b is any real number and n is a natural number, then

$$b^n = \overbrace{b \cdot b \cdot b \cdot \cdots \cdot b}^{b \text{ is a factor } n \text{ times}}$$

where b is the **base** and n is the **exponent.**

Example

$$\left(\frac{3}{4}\right)^3 = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

$$-5^4 = -(5 \cdot 5 \cdot 5 \cdot 5) = -625$$

$$(-5)^4 = (-5)(-5)(-5)(-5) = 625$$

Pay close attention to the difference between -5^4 (the base is 5) and $(-5)^4$ (the base is -5).

QUESTION What is the value of **a.** -2^5 and **b.** $(-2)^5$?

ANSWER **a.** $-2^5 = -(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2) = -32$
b. $(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32$

We can extend the definition of an exponent to all the integers. We first deal with the case of zero as an exponent.

Definition of b^0

For any nonzero real number b, $b^0 = 1$.

Example

$3^0 = 1 \qquad \left(\frac{3}{4}\right)^0 = 1 \qquad -7^0 = -1 \qquad (a^2 + 1)^0 = 1$

take note

Note that $-7^0 = -(7^0) = -1$.

Now we extend the definition to include negative integers.

Definition of b^{-n}

If $b \neq 0$ and n is a natural number, then $b^{-n} = \frac{1}{b^n}$ and $\frac{1}{b^{-n}} = b^n$.

Example

$3^{-2} = \frac{1}{3^2} = \frac{1}{9} \qquad \frac{1}{4^{-3}} = 4^3 = 64 \qquad \frac{5^{-2}}{7^{-1}} = \frac{7}{5^2} = \frac{7}{25}$

take note

Using the definition of b^{-n},

$$\frac{5^{-2}}{7^{-1}} = \frac{\frac{1}{5^2}}{\frac{1}{7}}$$

Using the rules for dividing fractions, we have

$$\frac{\frac{1}{5^2}}{\frac{1}{7}} = \frac{1}{5^2} \div \frac{1}{7} = \frac{1}{5^2} \cdot \frac{7}{1} = \frac{7}{5^2}$$

EXAMPLE 1 » Evaluate an Exponential Expression

Evaluate.

a. $(-2^4)(-3)^2$

b. $\frac{(-4)^{-3}}{(-2)^{-5}}$

c. $-\pi^0$

Solution

a. $(-2^4)(-3)^2 = -(2 \cdot 2 \cdot 2 \cdot 2)(-3)(-3) = -(16)(9) = -144$

b. $\frac{(-4)^{-3}}{(-2)^{-5}} = \frac{(-2)(-2)(-2)(-2)(-2)}{(-4)(-4)(-4)} = \frac{-32}{-64} = \frac{1}{2}$

c. $-\pi^0 = -(\pi^0) = -1$

» *Try Exercise 10, page 30*

Alternative to Example 1

Evaluate.

a. $6^{-3}(-3)^4$

■ $\frac{3}{8}$

b. -6^0

■ -1

take note

The expression in **c.** is similar to $-5^4 = -625$, which was discussed earlier.

When working with exponential expressions containing variables, we must ensure that a value of the variable does not result in an undefined expression. For instance, $x^{-2} = \frac{1}{x^2}$. Because division by zero is not allowed, for the expression x^{-2}, we must assume that $x \neq 0$. Therefore, to avoid problems with undefined expressions, we will use the following restriction agreement.

Restriction Agreement

The expressions 0^0, 0^n (where n is a negative integer), and $\frac{a}{0}$ are all undefined expressions. Therefore, all values of variables in this text are restricted to avoid any one of these expressions.

Example

In the expression $\frac{x^0y^{-3}}{z-4}$, $x \neq 0$, $y \neq 0$, and $z \neq 4$.

In the expression $\frac{(a-1)^0}{b+2}$, $a \neq 1$ and $b \neq -2$.

Exponential expressions containing variables are simplified using the following properties of exponents.

Properties of Exponents

If m, n, and p are integers and a and b are real numbers, then

Product $\quad b^m \cdot b^n = b^{m+n}$

Quotient $\quad \frac{b^m}{b^n} = b^{m-n}, \quad b \neq 0$

Power $\quad (b^m)^n = b^{mn} \quad (a^mb^n)^p = a^{mp}b^{np}$

$$\left(\frac{a^m}{b^n}\right)^p = \frac{a^{mp}}{b^{np}}, \quad b \neq 0$$

Example

$a^4 \cdot a \cdot a^3 = a^{4+1+3} = a^8$ • Add the exponents on the like bases. Recall that $a = a^1$.

$(x^4y^3)(xy^5z^2) = x^{4+1}y^{3+5}z^2 = x^5y^8z^2$ • Add the exponents on the like bases.

$\frac{a^7b}{a^2b^5} = a^{7-2}b^{1-5} = a^5b^{-4} = \frac{a^5}{b^4}$ • Subtract the exponents on the like bases.

$(uv^3)^5 = u^{1\cdot5}v^{3\cdot5} = u^5v^{15}$ • Multiply the exponents.

$\left(\frac{2x^5}{5y^4}\right)^3 = \frac{2^{1\cdot3}x^{5\cdot3}}{5^{1\cdot3}y^{4\cdot3}} = \frac{2^3x^{15}}{5^3y^{12}} = \frac{8x^{15}}{125y^{12}}$ • Multiply the exponents.

QUESTION Can the exponential expression x^5y^3 be simplified using the properties of exponents?

ANSWER No. The bases are not the same.

Integrating Technology

Exponential expressions such as a^{b^c} can be confusing. The generally accepted meaning of a^{b^c} is $a^{(b^c)}$. However, some graphing calculators do not evaluate exponential expressions in this way. Enter `2^3^4` in a graphing calculator. If the result is approximately 2.42×10^{24}, then the calculator evaluated $2^{(3^4)}$. If the result is 4096, then the calculator evaluated $(2^3)^4$. To ensure that you calculate the value you intend, we strongly urge you to use parentheses. For instance, entering `2^(3^4)` will produce 2.42×10^{24} and entering `(2^3)^4` will produce 4096.

To simplify an expression involving exponents, write the expression in a form in which *each base occurs at most once* and *no powers of powers or negative exponents occur.*

Alternative to Example 2

Simplify.

a. $(-3ab^4)(-6a^2b^{-4})$

■ $18a^3$

b. $-2ab^2(-3a^2b)^3$

■ $54a^7b^5$

c. $\left(\dfrac{5x^{-2}y^3}{15x^2y}\right)^{-1}$

■ $\dfrac{3x^4}{y^2}$

EXAMPLE 2 Simplify Exponential Expressions

Simplify.

a. $(5x^2y)(-4x^3y^5)$ **b.** $(3x^2yz^{-4})^3$ **c.** $\dfrac{-12x^5y}{18x^2y^6}$ **d.** $\left(\dfrac{4p^2q}{6pq^4}\right)^{-2}$

Solution

a. $(5x^2y)(-4x^3y^5) = [5(-4)]x^{2+3}y^{1+5}$

$= -20x^5y^6$

• Multiply the coefficients. Multiply the variables by adding the exponents on the like bases.

b. $(3x^2yz^{-4})^3 = 3^{1\cdot3}x^{2\cdot3}y^{1\cdot3}z^{-4\cdot3}$

• Use the power property of exponents.

$= 3^3x^6y^3z^{-12} = \dfrac{27x^6y^3}{z^{12}}$

c. $\dfrac{-12x^5y}{18x^2y^6} = -\dfrac{2}{3}x^{5-2}y^{1-6}$

• Simplify $\dfrac{-12}{18} = -\dfrac{2}{3}$. Divide the variables by subtracting exponents of like bases.

$= -\dfrac{2}{3}x^3y^{-5} = -\dfrac{2x^3}{3y^5}$

d. $\left(\dfrac{4p^2q}{6pq^4}\right)^{-2} = \left(\dfrac{2p^{2-1}q^{1-4}}{3}\right)^{-2} = \left(\dfrac{2pq^{-3}}{3}\right)^{-2}$

• Use the quotient property of exponents.

$= \dfrac{2^{1(-2)}p^{1(-2)}q^{-3(-2)}}{3^{1(-2)}} = \dfrac{2^{-2}p^{-2}q^6}{3^{-2}}$

• Use the power property of exponents.

$= \dfrac{9q^6}{4p^2}$

• Write the answer in simplest form.

Try Exercise 36, page 31

Math Matters

- Approximately 3.1×10^6 orchid seeds weigh 1 ounce.
- Computer scientists measure an operation in nanoseconds.

1 nanosecond $= 1 \times 10^{-9}$ second

- If a spaceship traveled 25,000 mph, it would require approximately 2.7×10^9 years to travel from one end of the universe to the other.

Scientific Notation

The exponent theorems provide a compact method of writing very large or very small numbers. The method is called *scientific notation.* A number written in **scientific notation** has the form $a \cdot 10^n$, where n is an integer and $1 \le a < 10$. The following procedure is used to change a number from its decimal form to scientific notation.

For numbers greater than 10, move the decimal point to the position to the right of the first digit. The exponent n will equal the number of places the decimal point has been moved. For example,

$$7{,}430{,}000 = 7.43 \times 10^6$$

6 places

For numbers less than 1, move the decimal point to the right of the first nonzero digit. The exponent n will be negative, and its absolute value will equal the number of places the decimal point has been moved. For example,

$$0.00000078 = 7.8 \times 10^{-7}$$

7 places

To change a number from scientific notation to its decimal form, reverse the procedure. That is, if the exponent is positive, move the decimal point to the right the same number of places as the exponent. For example,

$$3.5 \times 10^5 = 350{,}000$$

5 places

If the exponent is negative, move the decimal point to the left the same number of places as the absolute value of the exponent. For example,

$$2.51 \times 10^{-8} = 0.0000000251$$

8 places

Most scientific calculators display very large and very small numbers in scientific notation. The number $450{,}000^2$ is displayed as `2.025 E 11`. This means $450{,}000^2 = 2.025 \times 10^{11}$.

Alternative to Example 3

The approximate diameter of the Milky Way galaxy is 9.5×10^4 light-years. If one light-year is 5.7×10^{12} miles, what is the approximate diameter, in miles, of the Milky Way galaxy?

- **5.4×10^{17} miles**

EXAMPLE 3 Simplify an Expression Using Scientific Notation

The Andromeda galaxy is approximately 1.4×10^{19} miles from Earth. If a spacecraft could travel 2.8×10^{12} miles in 1 year (about one-half the speed of light), how many years would it take for the spacecraft to reach the Andromeda galaxy?

Solution

To find the time, divide the distance by the speed.

$$t = \frac{1.4 \times 10^{19}}{2.8 \times 10^{12}} = \frac{1.4}{2.8} \times 10^{19-12} = 0.5 \times 10^7 = 5.0 \times 10^6$$

It would take 5.0×10^6 (or 5,000,000) years for the spacecraft to reach the Andromeda galaxy.

Try Exercise 52, page 31

Rational Exponents and Radicals

To this point, the expression b^n has been defined for real numbers b and integers n. Now we wish to extend the definition of exponents to include rational numbers so that expressions such as $2^{1/2}$ will be meaningful. Not just any definition will do. We want a definition of rational exponents for which the properties of integer exponents are true. The following example shows the direction we can take to accomplish our goal.

If the product property for exponential expressions is to hold for rational exponents, then for rational numbers p and q, $b^p b^q = b^{p+q}$. For example,

$$9^{1/2} \cdot 9^{1/2} \quad \text{must equal} \quad 9^{1/2+1/2} = 9^1 = 9$$

Thus $9^{1/2}$ must be a square root of 9. That is, $9^{1/2} = 3$.

The example suggests that $b^{1/n}$ can be defined in terms of roots according to the following definition.

Definition of $b^{1/n}$

If n is an even positive integer and $b \geq 0$, then $b^{1/n}$ is the nonnegative real number such that $(b^{1/n})^n = b$.

If n is an odd positive integer, then $b^{1/n}$ is the real number such that $(b^{1/n})^n = b$.

Example

- $25^{1/2} = 5$ because $5^2 = 25$.
- $(-64)^{1/3} = -4$ because $(-4)^3 = -64$.
- $16^{1/2} = 4$ because $4^2 = 16$.
- $-16^{1/2} = -(16^{1/2}) = -4$.
- $(-16)^{1/2}$ is not a real number.
- $(-32)^{1/5} = -2$ because $(-2)^5 = -32$.

If n is an even positive integer and $b < 0$, then $b^{1/n}$ is a *complex number*. Complex numbers are discussed in Section P.6.

To define expressions such as $8^{2/3}$, we will extend our definition of exponents even further. Because we want the power property $(b^p)^q = b^{pq}$ to be true for rational exponents also, we must have $(b^{1/n})^m = b^{m/n}$. With this in mind, we make the following definition.

Definition of $b^{m/n}$

For all positive integers m and n such that m/n is in simplest form, and for all real numbers b for which $b^{1/n}$ is a real number,

$$b^{m/n} = (b^{1/n})^m = (b^m)^{1/n}$$

Because $b^{m/n}$ is defined as $(b^{1/n})^m$ and also as $(b^m)^{1/n}$, we can evaluate expressions such as $8^{4/3}$ in more than one way. For example, because $8^{1/3}$ is a real number, $8^{4/3}$ can be evaluated in either of the following ways:

$$8^{4/3} = (8^{1/3})^4 = 2^4 = 16$$
$$8^{4/3} = (8^4)^{1/3} = 4096^{1/3} = 16$$

Of the two methods, the $b^{m/n} = (b^{1/n})^m$ method is usually easier to apply, provided you can evaluate $b^{1/n}$.

Here are some additional examples.

$$64^{2/3} = (64^{1/3})^2 = 4^2 = 16$$
$$32^{-6/5} = \frac{1}{32^{6/5}} = \frac{1}{(32^{1/5})^6} = \frac{1}{2^6} = \frac{1}{64}$$
$$81^{0.75} = 81^{3/4} = (81^{1/4})^3 = 3^3 = 27$$

Integrating Technology

For the examples above, the base of the exponential expression is an integer power of the denominator of the exponent.

$$64 = 4^3 \qquad 32 = 2^5 \qquad 81 = 3^4$$

If the base of the exponential expression is not a power of the denominator of the exponent, a calculator is used to evaluate the expression. Some examples are shown at the left.

In each of these examples, the value of the exponential expression is an irrational number. The decimal display is only an approximation of the actual result.

```
16^(3/8)
                2.828427125
25^(-1/5)
                .5253055609
42^(.14)
                1.687543205
```

The following exponent properties were stated earlier, but they are restated here to remind you that they have now been extended to apply to rational exponents.

Properties of Rational Exponents

If p, q, and r represent rational numbers and a and b are positive real numbers, then

Product $b^p \cdot b^q = b^{p+q}$

Quotient $\dfrac{b^p}{b^q} = b^{p-q}$

Power $(b^p)^q = b^{pq}$ $\qquad (a^p b^q)^r = a^{pr} b^{qr}$

$\left(\dfrac{a^p}{b^q}\right)^r = \dfrac{a^{pr}}{b^{qr}} \qquad b^{-p} = \dfrac{1}{b^p}$

Recall that an exponential expression is in simplest form when no powers of powers or negative exponents occur and each base occurs at most once.

Alternative to Example 4

a. Simplify: $\left(-\dfrac{2x^2y}{3xy^3}\right)^{-2}$

■ $\dfrac{9y^4}{4x^2}$

b. Simplify: $(x^n)^2$

■ x^{2n}

EXAMPLE 4 » Simplify Exponential Expressions

Simplify: $\left(\dfrac{x^2y^3}{x^{-3}y^5}\right)^{1/2}$ (Assume $x > 0, y > 0$.)

Solution

$$\left(\frac{x^2y^3}{x^{-3}y^5}\right)^{1/2} = (x^{2-(-3)}y^{3-5})^{1/2} = (x^5y^{-2})^{1/2} = x^{5/2}y^{-1} = \frac{x^{5/2}}{y}$$

» *Try Exercise 68, page 31*

Simplify Radical Expressions

Radicals, expressed by the notation $\sqrt[n]{b}$, are also used to denote roots. The number b is the **radicand,** and the positive integer n is the **index** of the radical.

Math Matters

The formula for kinetic energy (energy of motion) that is used in Einstein's Theory of Relativity involves a radical.

$$\text{K.E}_r = mc^2\left(\frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}} - 1\right)$$

where m is the mass of the object at rest, v is the speed of the object, and c is the speed of light.

Definition of $\sqrt[n]{b}$

If n is a positive integer and b is a real number such that $b^{1/n}$ is a real number, then $\sqrt[n]{b} = b^{1/n}$.

If the index n equals 2, then the radical $\sqrt[2]{b}$ is written as simply $\sqrt{b}$, and it is referred to as the **principal square root of *b*** or simply the **square root of *b*.**

The symbol $\sqrt{b}$ is reserved to represent the nonnegative square root of b. To represent the negative square root of b, write $-\sqrt{b}$. For example, $\sqrt{25} = 5$, whereas $-\sqrt{25} = -5$.

Definition of $(\sqrt[n]{b})^m$

For all positive integers n, all integers m, and all real numbers b such that $\sqrt[n]{b}$ is a real number, $(\sqrt[n]{b})^m = \sqrt[n]{b^m} = b^{m/n}$.

When $\sqrt[n]{b}$ is a real number, the equations

$$b^{m/n} = \sqrt[n]{b^m} \quad \text{and} \quad b^{m/n} = (\sqrt[n]{b})^m$$

can be used to write exponential expressions such as $b^{m/n}$ in radical form. Use the denominator n as the index of the radical and the numerator m as the power of the radicand or as the power of the radical. For example,

$(5xy)^{2/3} = (\sqrt[3]{5xy})^2 = \sqrt[3]{25x^2y^2}$ • **Use the denominator 3 as the index of the radical and the numerator 2 as the power of the radical.**

The equations

$$b^{m/n} = \sqrt[n]{b^m} \qquad \text{and} \qquad b^{m/n} = (\sqrt[n]{b})^m$$

also can be used to write radical expressions in exponential form. For example,

$\sqrt{(2ab)^3} = (2ab)^{3/2}$ • Use the index 2 as the denominator of the power and the exponent 3 as the numerator of the power.

The definition of $(\sqrt[n]{b})^m$ often can be used to evaluate radical expressions. For instance,

$$(\sqrt[3]{8})^4 = 8^{4/3} = (8^{1/3})^4 = 2^4 = 16$$

Care must be exercised when simplifying even roots (square roots, fourth roots, sixth roots, and so on) of variable expressions. Consider $\sqrt{x^2}$ when $x = 5$ and when $x = -5$.

Case 1 If $x = 5$, then $\sqrt{x^2} = \sqrt{5^2} = \sqrt{25} = 5 = x$.

Case 2 If $x = -5$, then $\sqrt{x^2} = \sqrt{(-5)^2} = \sqrt{25} = 5 = -x$.

These two cases suggest that

$$\sqrt{x^2} = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

TO REVIEW

Absolute Value

See page 6.

Recalling the definition of absolute value, we can write this more compactly as $\sqrt{x^2} = |x|$.

Simplifying odd roots of a variable expression does not require using the absolute value symbol. Consider $\sqrt[3]{x^3}$ when $x = 5$ and when $x = -5$.

Case 1 If $x = 5$, then $\sqrt[3]{x^3} = \sqrt[3]{5^3} = \sqrt[3]{125} = 5 = x$.

Case 2 If $x = -5$, then $\sqrt[3]{x^3} = \sqrt[3]{(-5)^3} = \sqrt[3]{-125} = -5 = x$.

Thus $\sqrt[3]{x^3} = x$.

Although we have illustrated this principle only for square roots and cube roots, the same reasoning can be applied to other cases. The general result is given below.

Definition of $\sqrt[n]{b^n}$

If n is an even natural number and b is a real number, then

$$\sqrt[n]{b^n} = |b|$$

If n is an odd natural number and b is a real number, then

$$\sqrt[n]{b^n} = b$$

Example

$\sqrt[4]{16z^4} = 2|z| \qquad \sqrt[5]{32a^5} = 2a$

Because radicals are defined in terms of rational powers, the properties of radicals are similar to those of exponential expressions.

Properties of Radicals

If m and n are natural numbers and a and b are positive real numbers, then

Product $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$

Quotient $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Index $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

A radical is in **simplest form** if it meets all of the following criteria.

1. The radicand contains only powers less than the index. ($\sqrt{x^5}$ does not satisfy this requirement because 5, the exponent, is greater than 2, the index.)
2. The index of the radical is as small as possible. ($\sqrt[9]{x^3}$ does not satisfy this requirement because $\sqrt[9]{x^3} = x^{3/9} = x^{1/3} = \sqrt[3]{x}$.)
3. The denominator has been rationalized. That is, no radicals occur in the denominator. ($1/\sqrt{2}$ does not satisfy this requirement.)
4. No fractions occur under the radical sign. ($\sqrt[4]{2/x^3}$ does not satisfy this requirement.)

Radical expressions are simplified by using the properties of radicals. Here are some examples.

Alternative to Example 5

a. Simplify: $\sqrt{18x^2}$

■ $3|x|\sqrt{2}$

b. Simplify: $\sqrt[3]{-16x^4y^6}$

■ $-2xy^2\sqrt[3]{2x}$

EXAMPLE 5 Simplify Radical Expressions

Simplify.

a. $\sqrt[4]{32x^3y^4}$ b. $\sqrt[3]{162x^4y^6}$

Solution

a. $\sqrt[4]{32x^3y^4} = \sqrt[4]{2^5x^3y^4} = \sqrt[4]{(2^4y^4)\cdot(2x^3)}$ • Factor and group factors that can be written as a power of the index.

$= \sqrt[4]{2^4y^4}\cdot\sqrt[4]{2x^3}$ • Use the product property of radicals.

$= 2|y|\sqrt[4]{2x^3}$ • Recall that for n even, $\sqrt[n]{b^n} = |b|$.

b. $\sqrt[3]{162x^4y^6} = \sqrt[3]{(2\cdot 3^4)x^4y^6}$ • Factor and group factors that can be written as a power of the index.

$= \sqrt[3]{(3xy^2)^3\cdot(2\cdot 3x)}$

$= \sqrt[3]{(3xy^2)^3}\cdot\sqrt[3]{6x}$ • Use the product property of radicals.

$= 3xy^2\sqrt[3]{6x}$ • Recall that for n odd, $\sqrt[n]{b^n} = b$.

Try Exercise 84, page 31

Like radicals have the same radicand and the same index. For instance,

$$3\sqrt[3]{5xy^2} \quad \text{and} \quad -4\sqrt[3]{5xy^2}$$

are like radicals. Addition and subtraction of like radicals are accomplished by using the distributive property. For example,

$$4\sqrt{3x} - 9\sqrt{3x} = (4-9)\sqrt{3x} = -5\sqrt{3x}$$
$$2\sqrt[3]{y^2} + 4\sqrt[3]{y^2} - \sqrt[3]{y^2} = (2+4-1)\sqrt[3]{y^2} = 5\sqrt[3]{y^2}$$

The sum $2\sqrt{3} + 6\sqrt{5}$ cannot be simplified further because the radicands are not the same. The sum $3\sqrt[3]{x} + 5\sqrt[4]{x}$ cannot be simplified because the indices are not the same.

Sometimes it is possible to simplify radical expressions that do not appear to be like radicals by simplifying each radical expression.

Alternative to Example 6

a. Simplify: $2\sqrt{2x^3} + 4x\sqrt{8x}$. Assume $x \geq 0$.

■ $10x\sqrt{2x}$

b. Simplify: $2b\sqrt[3]{16b^2} + \sqrt[3]{128b^5}$

■ $8b\sqrt[3]{2b^2}$

EXAMPLE 6 Combine Radical Expressions

Simplify: $5x\sqrt[3]{16x^4} - \sqrt[3]{128x^7}$

Solution

$5x\sqrt[3]{16x^4} - \sqrt[3]{128x^7}$

$= 5x\sqrt[3]{2^4x^4} - \sqrt[3]{2^7x^7}$ • Factor.

$= 5x\sqrt[3]{2^3x^3}\cdot\sqrt[3]{2x} - \sqrt[3]{2^6x^6}\cdot\sqrt[3]{2x}$ • Group factors that can be written as a power of the index.

$= 5x(2x\sqrt[3]{2x}) - 2^2x^2\cdot\sqrt[3]{2x}$ • Use the product property of radicals.

$= 10x^2\sqrt[3]{2x} - 4x^2\sqrt[3]{2x}$ • Simplify.

$= 6x^2\sqrt[3]{2x}$

Try Exercise 92, page 31

Multiplication of radical expressions is accomplished by using the distributive property. For instance,

$\sqrt{5}(\sqrt{20} - 3\sqrt{15}) = \sqrt{5}(\sqrt{20}) - \sqrt{5}(3\sqrt{15})$ • Use the distributive property.

$= \sqrt{100} - 3\sqrt{75}$ • Multiply the radicals.

$= 10 - 3\cdot 5\sqrt{3}$ • Simplify.

$= 10 - 15\sqrt{3}$

Finding the product of more complicated radical expressions may require repeated use of the distributive property.

EXAMPLE 7 Multiply Radical Expressions

Perform the indicated operation:

$$(\sqrt{3}+5)(\sqrt{3}-2)$$

Alternative to Example 7

a. Simplify: $(2\sqrt{x} - 3)^2$

■ $4x - 12\sqrt{x} + 9$

b. Simplify: $(3\sqrt{x} + 4)(3\sqrt{x} - 4)$

■ $9x - 16$

Solution

$(\sqrt{3} + 5)(\sqrt{3} - 2)$

$= (\sqrt{3} + 5)\sqrt{3} - (\sqrt{3} + 5)2$ • Use the distributive property.

$= (\sqrt{3}\sqrt{3} + 5\sqrt{3}) - (2\sqrt{3} + 2 \cdot 5)$ • Use the distributive property.

$= 3 + 5\sqrt{3} - 2\sqrt{3} - 10$

$= -7 + 3\sqrt{3}$

Try Exercise 102, page 32

To **rationalize the denominator** of a fraction means to write the fraction in an equivalent form that does not involve any radicals in the denominator. This is accomplished by multiplying the numerator and denominator of the radical expression by an expression that will cause the radicand in the denominator to be a perfect root of the index.

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3^2}} = \frac{5\sqrt{3}}{3}$$

• Recall that $\sqrt{3}$ means $\sqrt[2]{3}$. Multiply numerator and denominator by $\sqrt{3}$ so that the radicand is a perfect root of the index of the radical.

$$\frac{2}{\sqrt[3]{7}} = \frac{2}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^2}} = \frac{2\sqrt[3]{7^2}}{\sqrt[3]{7^3}} = \frac{2\sqrt[3]{49}}{7}$$

• Multiply numerator and denominator by $\sqrt[3]{7^2}$ so that the radicand is a perfect root of the index of the radical.

$$\frac{5}{\sqrt[4]{x^5}} = \frac{5}{\sqrt[4]{x^5}} \cdot \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = \frac{5\sqrt[4]{x^3}}{\sqrt[4]{x^8}} = \frac{5\sqrt[4]{x^3}}{x^2}$$

• Multiply numerator and denominator by $\sqrt[4]{x^3}$ so that the radicand is a perfect root of the index of the radical.

Alternative to Example 8

a. Simplify: $\sqrt{\dfrac{5x}{10y}}$

■ $\dfrac{\sqrt{2xy}}{2y}$

b. Simplify: $\dfrac{6}{\sqrt[3]{9}}$

■ $2\sqrt[3]{3}$

c. Simplify: $\dfrac{a^2}{\sqrt[3]{a}}$

■ $a\sqrt[3]{a^2}$

EXAMPLE 8 Rationalize the Denominator

Rationalize the denominator. **a.** $\dfrac{5}{\sqrt[3]{a}}$ **b.** $\sqrt{\dfrac{3}{32y}}$

Solution

a. $\dfrac{5}{\sqrt[3]{a}} = \dfrac{5}{\sqrt[3]{a}} \cdot \dfrac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}} = \dfrac{5\sqrt[3]{a^2}}{\sqrt[3]{a^3}} = \dfrac{5\sqrt[3]{a^2}}{a}$ • Use $\sqrt[3]{a} \cdot \sqrt[3]{a^2} = \sqrt[3]{a^3} = a$.

b. $\sqrt{\dfrac{3}{32y}} = \dfrac{\sqrt{3}}{\sqrt{32y}} = \dfrac{\sqrt{3}}{4\sqrt{2y}} = \dfrac{\sqrt{3}}{4\sqrt{2y}} \cdot \dfrac{\sqrt{2y}}{\sqrt{2y}} = \dfrac{\sqrt{6y}}{8y}$

Try Exercise 112, page 32

To rationalize the denominator of a fractional expression such as

$$\frac{1}{\sqrt{m} + \sqrt{n}}$$

we make use of the conjugate of $\sqrt{m} + \sqrt{n}$, which is $\sqrt{m} - \sqrt{n}$. The product of these conjugate pairs does not involve a radical.

$$(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) = m - n$$

Alternative to Example 7

a. Simplify: $(2\sqrt{x} - 3)^2$

■ $4x - 12\sqrt{x} + 9$

b. Simplify: $(3\sqrt{x} + 4)(3\sqrt{x} - 4)$

■ $9x - 16$

Solution

$$(\sqrt{3} + 5)(\sqrt{3} - 2)$$
$$= (\sqrt{3} + 5)\sqrt{3} - (\sqrt{3} + 5)2$$ • Use the distributive property.
$$= (\sqrt{3}\sqrt{3} + 5\sqrt{3}) - (2\sqrt{3} + 2 \cdot 5)$$ • Use the distributive property.
$$= 3 + 5\sqrt{3} - 2\sqrt{3} - 10$$
$$= -7 + 3\sqrt{3}$$

Try Exercise 102, page 32

To **rationalize the denominator** of a fraction means to write the fraction in an equivalent form that does not involve any radicals in the denominator. This is accomplished by multiplying the numerator and denominator of the radical expression by an expression that will cause the radicand in the denominator to be a perfect root of the index.

$$\frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{\sqrt{3^2}} = \frac{5\sqrt{3}}{3}$$

• Recall that $\sqrt{3}$ means $\sqrt[2]{3}$. Multiply numerator and denominator by $\sqrt{3}$ so that the radicand is a perfect root of the index of the radical.

$$\frac{2}{\sqrt[3]{7}} = \frac{2}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7^2}}{\sqrt[3]{7^2}} = \frac{2\sqrt[3]{7^2}}{\sqrt[3]{7^3}} = \frac{2\sqrt[3]{49}}{7}$$

• Multiply numerator and denominator by $\sqrt[3]{7^2}$ so that the radicand is a perfect root of the index of the radical.

$$\frac{5}{\sqrt[4]{x^5}} = \frac{5}{\sqrt[4]{x^5}} \cdot \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = \frac{5\sqrt[4]{x^3}}{\sqrt[4]{x^8}} = \frac{5\sqrt[4]{x^3}}{x^2}$$

• Multiply numerator and denominator by $\sqrt[4]{x^3}$ so that the radicand is a perfect root of the index of the radical.

Alternative to Example 8

a. Simplify: $\sqrt{\frac{5x}{10y}}$

■ $\frac{\sqrt{2xy}}{2y}$

b. Simplify: $\frac{6}{\sqrt[3]{9}}$

■ $2\sqrt[3]{3}$

c. Simplify: $\frac{a^2}{\sqrt[3]{a}}$

■ $a\sqrt[3]{a^2}$

EXAMPLE 8 Rationalize the Denominator

Rationalize the denominator. **a.** $\frac{5}{\sqrt[3]{a}}$ **b.** $\sqrt{\frac{3}{32y}}$

Solution

a. $$\frac{5}{\sqrt[3]{a}} = \frac{5}{\sqrt[3]{a}} \cdot \frac{\sqrt[3]{a^2}}{\sqrt[3]{a^2}} = \frac{5\sqrt[3]{a^2}}{\sqrt[3]{a^3}} = \frac{5\sqrt[3]{a^2}}{a}$$

• Use $\sqrt[3]{a} \cdot \sqrt[3]{a^2} = \sqrt[3]{a^3} = a$.

b. $$\sqrt{\frac{3}{32y}} = \frac{\sqrt{3}}{\sqrt{32y}} = \frac{\sqrt{3}}{4\sqrt{2y}} = \frac{\sqrt{3}}{4\sqrt{2y}} \cdot \frac{\sqrt{2y}}{\sqrt{2y}} = \frac{\sqrt{6y}}{8y}$$

Try Exercise 112, page 32

To rationalize the denominator of a fractional expression such as

$$\frac{1}{\sqrt{m} + \sqrt{n}}$$

we make use of the conjugate of $\sqrt{m} + \sqrt{n}$, which is $\sqrt{m} - \sqrt{n}$. The product of these conjugate pairs does not involve a radical.

$$(\sqrt{m} + \sqrt{n})(\sqrt{m} - \sqrt{n}) = m - n$$

In Example 9 we use the conjugate of the denominator to rationalize the denominator.

Alternative to Example 9

a. Simplify: $\frac{3}{2-\sqrt{7}}$

■ $-(2+\sqrt{7})$

b. Simplify: $\frac{\sqrt{x}}{\sqrt{x}+2}$

■ $\frac{x-2\sqrt{x}}{x-4}$

EXAMPLE 9 Rationalize the Denominator

Rationalize the denominator.

a. $\frac{2}{\sqrt{3}+\sqrt{2}}$ **b.** $\frac{a+\sqrt{5}}{a-\sqrt{5}}$

Solution

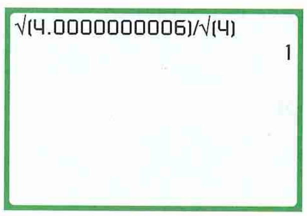
a. $$\frac{2}{\sqrt{3}+\sqrt{2}} = \frac{2}{\sqrt{3}+\sqrt{2}} \cdot \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{2\sqrt{3}-2\sqrt{2}}{3-2} = 2\sqrt{3}-2\sqrt{2}$$

b. $$\frac{a+\sqrt{5}}{a-\sqrt{5}} = \frac{a+\sqrt{5}}{a-\sqrt{5}} \cdot \frac{a+\sqrt{5}}{a+\sqrt{5}} = \frac{a^2+2a\sqrt{5}+5}{a^2-5}$$

Try Exercise 116, page 32

Topics for Discussion

1. Given that a is a real number, discuss when the expression $a^{p/q}$ represents a real number.
2. The expressions $-a^n$ and $(-a)^n$ do not always represent the same number. Discuss the situations in which the two expressions are equal and those in which they are not equal.
3. The calculator screen at the left shows the value of the quotient of two radical expressions. Is the answer correct? Explain what happened.



4. If you enter the expression for $\sqrt{5}$ on your calculator, the calculator will respond with 2.236067977 or some number close to that. Is this the exact value of $\sqrt{5}$? Is it possible to find the exact decimal value of $\sqrt{5}$ with a calculator? with a computer?

—Suggested Assignment: Exercises 1–117, every other odd; 119–131, odd; 132, 134, 139, 141.

Exercise Set P.2

In Exercises 1 to 12, evaluate each expression.

1. -5^3 -125

2. $(-5)^3$ -125

3. $\left(\frac{2}{3}\right)^0$ 1

4. -6^0 -1

5. 4^{-2} $\frac{1}{16}$

6. 3^{-4} $\frac{1}{81}$

7. $\frac{1}{2^{-5}}$ 32

8. $\frac{1}{3^{-3}}$ 27

9. $\frac{2^{-3}}{6^{-3}}$ 27

10. $\frac{4^{-2}}{2^{-3}}$ $\frac{1}{2}$

11. $-2x^0$ -2

12. $\frac{x^0}{4}$ $\frac{1}{4}$

In Exercises 13 to 38, write the exponential expression in simplest form.

13. $2x^{-4}$ $\frac{2}{x^4}$

14. $3y^{-2}$ $\frac{3}{y^2}$

15. $\frac{5}{z^{-6}}$ $5z^6$

16. $\frac{8}{x^{-5}}$ $8x^5$

17. $(x^3y^2)(xy^5)$ x^4y^7

18. $(uv^6)(u^2v)$ u^3v^7

19. $(-2ab^4)(-3a^2b^5)$ $6a^3b^9$

20. $(9xy^2)(-2x^2y^5)$ $-18x^3y^7$

21. $\frac{16a^7}{2a}$ $8a^6$

22. $\frac{24z^8}{-3z^3}$ $-8z^5$

23. $\frac{6a^4}{8a^8}$ $\frac{3}{4a^4}$

24. $\frac{12x^3}{16x^4}$ $\frac{3}{4x}$

25. $\frac{12x^3y^4}{18x^5y^2}$ $\frac{2y^2}{3x^2}$

26. $\frac{5v^4w^{-3}}{10v^8}$ $\frac{1}{2v^4w^3}$

27. $\frac{36a^{-2}b^3}{3ab^4}$ $\frac{12}{a^3b}$

28. $\frac{-48ab^{10}}{-32a^4b^3}$ $\frac{3b^7}{2a^3}$

29. $(-2m^3n^2)(-3mn^2)^2$ $-18m^5n^6$

30. $(2a^3b^2)^3(-4a^4b^2)$ $-32a^{13}b^8$

31. $(x^{-2}y)^2(xy)^{-2}$ $\frac{1}{x^6}$

32. $(x^{-1}y^2)^{-3}(x^2y^{-4})^{-3}$ $\frac{y^6}{x^3}$

33. $\left(\frac{3a^2b^3}{6a^4b^4}\right)^2$ $\frac{1}{4a^4b^2}$

34. $\left(\frac{2ab^2c^3}{5ab^2}\right)^3$ $\frac{8c^9}{125}$

35. $\frac{(-4x^2y^3)^2}{(2xy^2)^3}$ $2x$

36. $\frac{(-3a^2b^3)^2}{(-2ab^4)^3}$ $-\frac{9a}{8b^6}$

37. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^2$ $\frac{b^{10}}{a^{10}}$

38. $\left(\frac{x^{-3}y^{-4}}{x^{-2}y}\right)^{-2}$ x^2y^{10}

In Exercises 39 to 42, write the number in scientific notation.

39. 2,011,000,000,000 2.011×10^{12}

40. 49,100,000,000 4.91×10^{10}

41. 0.000000000562 5.62×10^{-10}

42. 0.000000402 4.02×10^{-7}

In Exercises 43 to 46, change the number from scientific notation to decimal notation.

43. 3.14×10^7 31,400,000

44. 4.03×10^9 4,030,000,000

45. -2.3×10^{-6} −0.0000023

46. 6.14×10^{-8} 0.0000000614

In Exercises 47 to 54, perform the indicated operation and write the answer in scientific notation.

47. $(3 \times 10^{12})(9 \times 10^{-5})$ 2.7×10^8

48. $(8.9 \times 10^{-5})(3.4 \times 10^{-6})$ 3.026×10^{-10}

49. $\frac{9 \times 10^{-3}}{6 \times 10^8}$ 1.5×10^{-11}

50. $\frac{2.5 \times 10^8}{5 \times 10^{10}}$ 5×10^{-3}

51. $\frac{(3.2 \times 10^{-11})(2.7 \times 10^{18})}{1.2 \times 10^{-5}}$ 7.2×10^{12}

52. $\frac{(6.9 \times 10^{27})(8.2 \times 10^{-13})}{4.1 \times 10^{15}}$ 1.38×10^0

53. $\frac{(4.0 \times 10^{-9})(8.4 \times 10^5)}{(3.0 \times 10^{-6})(1.4 \times 10^{18})}$ 8×10^{-16}

54. $\frac{(7.2 \times 10^8)(3.9 \times 10^{-7})}{(2.6 \times 10^{-10})(1.8 \times 10^{-8})}$ 6×10^{19}

In Exercises 55 to 76, evaluate each exponential expression.

55. $4^{3/2}$ 8

56. $-16^{3/2}$ -64

57. $-64^{2/3}$ -16

58. $125^{4/3}$ 625

59. $9^{-3/2}$ $\frac{1}{27}$

60. $32^{-3/5}$ $\frac{1}{8}$

61. $\left(\frac{4}{9}\right)^{1/2}$ $\frac{2}{3}$

62. $\left(\frac{16}{25}\right)^{3/2}$ $\frac{64}{125}$

63. $\left(\frac{1}{8}\right)^{-4/3}$ 16

64. $\left(\frac{8}{27}\right)^{-2/3}$ $\frac{9}{4}$

65. $(4a^{2/3}b^{1/2})(2a^{1/3}b^{3/2})$ $8ab^2$

66. $(6a^{3/5}b^{1/4})(-3a^{1/5}b^{3/4})$ $-18a^{4/5}b$

67. $(-3x^{2/3})(4x^{1/4})$ $-12x^{11/12}$

68. $(-5x^{1/3})(-4x^{1/2})$ $20x^{5/6}$

69. $(81x^8y^{12})^{1/4}$ $3x^2y^3$

70. $(27x^3y^6)^{2/3}$ $9x^2y^4$

71. $\frac{16z^{3/5}}{12z^{1/5}}$ $\frac{4z^{2/5}}{3}$

72. $\frac{6a^{2/3}}{9a^{1/3}}$ $\frac{2a^{1/3}}{3}$

73. $(2x^{2/3}y^{1/2})(3x^{1/6}y^{1/3})$ $6x^{5/6}y^{5/6}$

74. $\frac{x^{1/3}y^{5/6}}{x^{2/3}y^{1/6}}$ $\frac{y^{2/3}}{x^{1/3}}$

75. $\frac{9a^{3/4}b}{3a^{2/3}b^2}$ $\frac{3a^{1/12}}{b}$

76. $\frac{12x^{1/6}y^{1/4}}{16x^{3/4}y^{1/2}}$ $\frac{3}{4x^{7/12}y^{1/4}}$

In Exercises 77 to 86, simplify each radical expression.

77. $\sqrt{45}$ $3\sqrt{5}$

78. $\sqrt{75}$ $5\sqrt{3}$

79. $\sqrt[3]{24}$ $2\sqrt[3]{3}$

80. $\sqrt[3]{135}$ $3\sqrt[3]{5}$

81. $\sqrt[3]{-135}$ $-3\sqrt[3]{5}$

82. $\sqrt[3]{-250}$ $-5\sqrt[3]{2}$

83. $\sqrt{24x^2y^3}$ $2|x|y\sqrt{6y}$

84. $\sqrt{18x^2y^5}$ $3|x|y^2\sqrt{2y}$

85. $\sqrt[3]{16a^3y^7}$ $2ay^2\sqrt[3]{2y}$

86. $\sqrt[3]{54m^2n^7}$ $3n^2\sqrt[3]{2m^2n}$

In Exercises 87 to 94, simplify each radical and then combine like radicals.

87. $2\sqrt{32} - 3\sqrt{98}$ $-13\sqrt{2}$

88. $5\sqrt[3]{32} + 2\sqrt[3]{108}$ $16\sqrt[3]{4}$

89. $-8\sqrt[4]{48} + 2\sqrt[4]{243}$ $-10\sqrt[4]{3}$

90. $2\sqrt[3]{40} - 3\sqrt[3]{135}$ $-5\sqrt[3]{5}$

91. $4\sqrt[3]{32y^4} + 3y\sqrt[3]{108y}$ $17y\sqrt[3]{4y}$

92. $-3x\sqrt[3]{54x^4} + 2\sqrt[3]{16x^7}$ $-5x^2\sqrt[3]{2x}$

93. $x\sqrt[3]{8x^3y^4} - 4y\sqrt[3]{64x^6y}$ $-14x^2y\sqrt[3]{y}$

94. $4\sqrt{a^5b} - a^2\sqrt{ab}$ $3a^2\sqrt{ab}$

In Exercises 95 to 104, find the indicated product and express each result in simplest form.

95. $(\sqrt{5}+3)(\sqrt{5}+4)$ $17+7\sqrt{5}$

96. $(\sqrt{7}+2)(\sqrt{7}-5)$ $-3-3\sqrt{7}$

97. $(\sqrt{2}-3)(\sqrt{2}+3)$ -7

98. $(2\sqrt{7}+3)(2\sqrt{7}-3)$ 19

99. $(3\sqrt{z}-2)(4\sqrt{z}+3)$ $12z+\sqrt{z}-6$

100. $(4\sqrt{a}-\sqrt{b})(3\sqrt{a}+2\sqrt{b})$ $12a+5\sqrt{ab}-2b$

101. $(\sqrt{x}+2)^2$ $x+4\sqrt{x}+4$

102. $(3\sqrt{5y}-4)^2$ $45y-24\sqrt{5y}+16$

103. $(\sqrt{x-3}+2)^2$ $x+4\sqrt{x-3}+1$

104. $(\sqrt{2x+1}-3)^2$ $2x-6\sqrt{2x+1}+10$

In Exercises 105 to 118, simplify each expression by rationalizing the denominator. Write the result in simplest form.

105. $\dfrac{2}{\sqrt{2}}$ $\sqrt{2}$

106. $\dfrac{3x}{\sqrt{3}}$ $x\sqrt{3}$

107. $\sqrt{\dfrac{5}{18}}$ $\dfrac{\sqrt{10}}{6}$

108. $\sqrt{\dfrac{7}{40}}$ $\dfrac{\sqrt{70}}{20}$

109. $\dfrac{3}{\sqrt[3]{2}}$ $\dfrac{3\sqrt[3]{4}}{2}$

110. $\dfrac{2}{\sqrt[3]{4}}$ $\sqrt[3]{2}$

111. $\dfrac{4}{\sqrt[3]{8x^2}}$ $\dfrac{2\sqrt[3]{x}}{x}$

112. $\dfrac{2}{\sqrt[4]{4y}}$ $\dfrac{\sqrt[4]{4y^3}}{y}$

113. $\dfrac{3}{\sqrt{3}+4}$ $-\dfrac{3\sqrt{3}-12}{13}$

114. $\dfrac{2}{\sqrt{5}-2}$ $2\sqrt{5}+4$

115. $\dfrac{6}{2\sqrt{5}+2}$ $\dfrac{3\sqrt{5}-3}{4}$

116. $\dfrac{-7}{3\sqrt{2}-5}$ $3\sqrt{2}+5$

117. $\dfrac{3}{\sqrt{5}+\sqrt{x}}$ $\dfrac{3\sqrt{5}-3\sqrt{x}}{5-x}$

118. $\dfrac{5}{\sqrt{y}-\sqrt{3}}$ $\dfrac{5\sqrt{y}+5\sqrt{3}}{y-3}$

119. **National Debt** In December of 2005, the U.S. national debt was approximately 8.1×10^{12} dollars. At that time, the population of the U.S. was 2.98×10^8 people. In December of 2005, what was the U.S. debt per person? $\approx 2.72 \times 10^4$ dollars

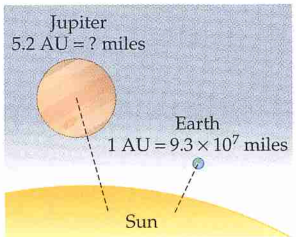
120. **Biology** The weight of one *E. coli* baterium is approximately 670 femtograms, where 1 femtogram $= 1 \times 10^{-15}$ gram. If one *E. coli* baterium can divide into two bacteria every 20 minutes, then after 24 hours there would be (assuming all bacteria survived) approximately 4.7×10^{21} bacteria. What is the weight, in grams, of these bacteria? 3.149×10^9 g

121. **Weight of an Orchid Seed** An orchid seed weighs approximately 3.2×10^{-8} ounce. If a package of seeds contains 1 ounce of orchid seeds, how many seeds are in the package? $\approx 3.13 \times 10^7$ seeds

122. **Laser Wavelength** The wavelength of a certain helium-neon laser is 800 nanometers. (1 nanometer is 1×10^{-9} meter.) The frequency, in cycles per second, of this wave is $\dfrac{1}{\text{wavelength}}$. What is the frequency of this laser? 1.25×10^6 cycles per second

123. **Doppler Effect** Astronomers can approximate the distance to a galaxy by measuring its *red shift*, which is a shift in the wavelength of light due to the velocity of the galaxy. This is similar to the way the sound of a siren coming toward you seems to have a higher pitch than the sound of the siren moving away from you. A formula for red shift is red shift $= \dfrac{\lambda_r - \lambda_s}{\lambda_s}$, where λ_r and λ_s are wavelengths of a certain frequency of light. Calculate the red shift for a galaxy for which $\lambda_r = 5.13 \times 10^{-7}$ meter and $\lambda_s = 5.06 \times 10^{-7}$ meter. $\approx 1.38 \times 10^{-2}$

124. **Astronomical Unit** Earth's mean distance from the sun is 9.3×10^7 miles. This distance is called the *astronomical unit* (AU). Jupiter is 5.2 AU from the sun. Find the distance in miles from Jupiter to the sun. $\approx 4.84 \times 10^8$ mi



125. **Astronomy** The sun is approximately 1.44×10^{11} meters from Earth. If light travels 3×10^8 meters per second, how many minutes does it take light from the sun to reach Earth? 8 min

126. **Mass of an Atom** One gram of hydrogen contains 6.023×10^{23} atoms. Find the mass of one hydrogen atom. $\approx 1.66 \times 10^{-24}$ g

127. **Cellular Phone Production** An electronics firm estimates that the revenue R it will receive from the sale of x cell phones (in thousands) can be approximated by $R = 1250x(2^{-0.007x})$. What is the estimated revenue when the company sells 20,000 cell phones? Round to the nearest dollar. \$22,688

128. **Drug Potency** The amount A (in milligrams) of digoxin, a drug taken by cardiac patients, remaining in the blood t hours after a patient takes a 2-milligram dose is given by $A = 2(10^{-0.0078t})$.

a. How much digoxin remains in the blood of a patient 4 hours after taking a 2-milligram dose? $\approx$1.86 mg

b. Suppose that a patient takes a 2-milligram dose of digoxin at 1:00 P.M. and another 2-milligram dose at 5:00 P.M. How much digoxin remains in the patient's blood at 6:00 P.M.? $\approx$3.79 mg

129. **WORLD POPULATION** An estimate of the world's future population P is given by $P = 6.5(2^{0.016354n})$, where n is the number of years after 2005 and P is in billions. Using this estimate, what will the world's population be in 2050? $\approx$10.8 billion

130. **LEARNING THEORY** In a psychology experiment, students were given a nine-digit number to memorize. The percent P of students who remembered the number t minutes after it was read to them can be given by $P = 90 - 3t^{2/3}$. What percent of the students remembered the number after 1 hour? $\approx$44.02%

131. **OCEANOGRAPHY** The percent P of light that will pass to a depth d, in meters, at a certain place in the ocean is given by $P = 10^{2-(d/40)}$. Find, to the nearest percent, the amount of light that will pass to a depth of **a.** 10 meters and **b.** 25 meters below the surface of the ocean.
a. 56% **b.** 24%

Connecting Concepts

132. If $2^x = y$, then find 2^{x-4} in terms of y. $\dfrac{y}{2^4}$

133. If a and b are nonzero numbers and $a < b$, is the statement $a^{-1} < b^{-1}$ a true statement? Give a reason for your answer. No. $2 < 3$, but $\dfrac{1}{2} > \dfrac{1}{3}$.

134. How many digits are in the product $4^{50} \cdot 5^{100}$? 101 digits

In Exercises 135 to 138, find the value of p for which the statement is true.

135. $a^{2/5}a^p = a^2$ $\quad \dfrac{8}{5}$

136. $b^{-3/4}b^{2p} = b^3$ $\quad \dfrac{15}{8}$

137. $\dfrac{x^{-3/4}}{x^{3p}} = x^4$ $\quad -\dfrac{19}{12}$

138. $(x^4x^{2p})^{1/2} = x$ $\quad -1$

In Exercises 139 to 143, rationalize the numerator.

139. $\dfrac{\sqrt{4+h}-2}{h}$ $\quad \dfrac{1}{\sqrt{4+h}+2}$

140. $\dfrac{\sqrt{9+h}-3}{h}$ $\quad \dfrac{1}{\sqrt{9+h}+3}$

141. $\sqrt{n^2+1}-n$ $\left(\textit{Hint: } \sqrt{n^2+1}-n = \dfrac{\sqrt{n^2+1}-n}{1}\right)$ $\quad \dfrac{1}{\sqrt{n^2+1}+n}$

142. $\sqrt{n^2+n}-n$ $\left(\textit{Hint: } \sqrt{n^2+n}-n = \dfrac{\sqrt{n^2+n}-n}{1}\right)$ $\quad \dfrac{n}{\sqrt{n^2+n}+n}$

143. Evaluate: $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$ $\quad 2$

Projects

1. **RELATIVITY THEORY** A moving object has energy, called *kinetic energy*, by virtue of its motion. As mentioned earlier in this chapter, the Theory of Relativity uses the following formula for kinetic energy.

$$\text{K.E}_r = mc^2\left(\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} - 1\right)$$

When the speed of an object is much less than the speed of light (3.0×10^8 meters per second) the formula

$$\text{K.E}_n = \frac{1}{2}mv^2$$

is used. In each formula, v is the velocity of the object in meters per second, m is its rest mass in kilograms, and c is the speed of light given above. In **a.** through **e.** calculate the percent error for each of the given velocities. The formula for percent error is

$$\%\text{ error} = \frac{|\text{K.E}_r - \text{K.E}_n|}{\text{K.E}_r} \times 100$$

a. $v = 30$ meters per second (speeding car on an expressway)

b. $v = 240$ meters per second (speed of a commercial jet)

Continued ▶

c. $v = 3.0 \times 10^7$ meters per second (10% of the speed of light)

d. $v = 1.5 \times 10^8$ meters per second (50% of the speed of light)

e. $v = 2.7 \times 10^8$ meters per second (90% of the speed of light)

f. Use your answers from **a.** through **e.** to give a reason why the formula for kinetic energy given by $K.E_n$ is adequate for most of our common experiences involving motion (walking, running, bicycling, driving, flying).

g. According to relativity theory, the mass m of an object changes as its velocity according to

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c_2}}}$$

where m_0 is the rest mass of the object. The approximate rest mass of an electron is 9.11×10^{-31} kilogram. What is the percent change, from its rest mass, in the mass of an electron that is traveling at $0.99c$ (99% of the speed of light)?

h. According to the Theory of Relativity, a particle (such as an electron or a spacecraft) cannot exceed the speed of light. Explain why the equation for $K.E_r$ suggests such a conclusion.

Section P.3

- Operations on Polynomials
- Applications of Polynomials

Polynomials

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A2.

PS1. Simplify: $-3(2a - 4b)$ [P.1] $-6a + 12b$

PS2. Simplify: $5 - 2(2x - 7)$ [P.1] $-4x + 19$

PS3. Simplify: $2x^2 + 3x - 5 + x^2 - 6x - 1$ [P.1] $3x^2 - 3x - 6$

PS4. Simplify: $4x^2 - 6x - 1 - 5x^2 + x$ [P.1] $-x^2 - 5x - 1$

PS5. True or false: $4 - 3x - 2x^2 = 2x^2 - 3x + 4$ [P.1] False

PS6. True or false: $\frac{12 + 15}{4} = \frac{\overset{3}{\cancel{12}} + 15}{\cancel{4}} = 18$ [P.1] False

Operations on Polynomials

A **monomial** is a constant, a variable, or the product of a constant and one or more variables, with the variables having only *nonnegative* integer exponents.

-8	z	$7y$	$-12a^2bc^3$
A number	**A variable**	**The product of a constant and one variable**	**The product of a constant and several variables**

The expression $3x^{-2}$ is *not* a monomial because it is the product of a constant and a variable with a *negative* integer exponent.

The constant multiplying the variables is called the **numerical coefficient** or **coefficient**. For $7y$, the coefficient is 7; for $-12a^2bc^3$, the coefficient is -12. The coefficient of z is 1 because $z = 1 \cdot z$. Similarly, the coefficient of $-x$ is -1 because $-x = -1 \cdot x$.

The **degree of a monomial** is the sum of the exponents of the variables. The degree of a nonzero constant is 0. The constant zero has no degree.

$7y$	**Degree is 1 because $y = y^1$.**
$-12a^2bc^3$	**Degree is $2 + 1 + 3 = 6$.**
-8	**Degree is 0.**

A **polynomial** is the sum of a finite number of monomials. Each monomial is called a **term** of the polynomial. The **degree of a polynomial** is the greatest of the degrees of the terms.

> ***take note***
> Note that the sign of a term is the sign that precedes the term.

Polynomial	Terms	Degree
$5x^4 - 6x^3 + 5x^2 - 7x - 8$	$5x^4, -6x^3, 5x^2, -7x, -8$	4
$-3xy^2 - 8xy + 6x$	$-3xy^2, -8xy, 6x$	3

Terms that have exactly the same variables raised to the same powers are called **like terms.** For example, $14x^2$ and $-x^2$ are like terms. $7x^2y$ and $5yx^2$ are like terms; the order of the variables is not important. The terms $6xy^2$ and $6x^2y$ are not like terms; the exponents on the variables are different.

A polynomial is said to be in simplest form if all its like terms have been combined. For example, the simplified form of $4x^2 + 3x + 5x - x^2$ is $3x^2 + 8x$. A **binomial** is a simplified polynomial with two terms; $3x^4 - 7$, $2xy - y^2$, and $x + 1$ are binomials. A **trinomial** is a simplified polynomial with three terms; $3x^2 + 6x - 1$, $2x^2 - 3xy + 7y^2$, and $x + y + 2$ are trinomials. A nonzero constant, such as 5, is called a **constant polynomial.**

Definition of the Standard Form of a Polynomial

The **standard form of a polynomial** of degree n in the variable x is

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n \neq 0$ and n is a nonnegative integer. The coefficient a_n is the **leading coefficient,** and a_0 is the **constant term.**

Example

Polynomial	Standard Form	Leading Coefficient
$6x - 7 + 2x^3$	$2x^3 + 6x - 7$	2
$4z^3 - 2z^4 + 3z - 9$	$-2z^4 + 4z^3 + 3z - 9$	-2
$y^5 - 3y^3 + 1 - 2y - y^2$	$y^5 - 3y^3 - y^2 - 2y + 1$	1

Alternative to Example 1
Write the polynomial $5x^3 - x^4 + 2x^2 - 7x - 8$ in standard form. Identify the degree, terms, constant term, leading coefficient, and coefficients of the polynomial.
■ $-x^4 + 5x^3 + 2x^2 - 7x - 8$; 4; $-x^4$, $5x^3$, $2x^2$, $-7x$, -8; -8; -1; $-1, 5, 2, -7, -8$

EXAMPLE 1 ⟫ Identify Terms Related to a Polynomial

Write the polynomial $6x^3 - x + 5 - 2x^4$ in standard form. Identify the degree, terms, constant term, leading coefficient, and coefficients of the polynomial.

Solution

A polynomial is in standard form when the terms are written in decreasing powers of the variable. The standard form of the polynomial is $-2x^4 + 6x^3 - x + 5$. In this form, the degree is 4; the terms are $-2x^4$, $6x^3$, $-x$, and 5; the constant term is 5. The leading coefficient is -2; the coefficients are $-2, 6, -1$, and 5.

⟫ *Try Exercise 12, page 40*

To add polynomials, add the coefficients of the like terms.

Alternative to Example 2
Add:
$(2x^2 - 6x + 7) + (4x^3 - 2x^2 + 8x - 3)$
■ $4x^3 + 2x + 4$

EXAMPLE 2 Add Polynomials

Add: $(3x^3 - 2x^2 - 6) + (4x^2 - 6x - 7)$

Solution

$$(3x^3 - 2x^2 - 6) + (4x^2 - 6x - 7)$$
$$= 3x^3 + (-2x^2 + 4x^2) + (-6x) + [(-6) + (-7)]$$
$$= 3x^3 + 2x^2 - 6x - 13$$

Try Exercise 24, page 40

The **additive inverse of the polynomial** $3x - 7$ is

$$-(3x - 7) = -3x + 7$$

QUESTION What is the additive inverse of $3x^2 - 8x + 7$?

To subtract a polynomial, we add its additive inverse. For example,

$$(2x - 5) - (3x - 7) = (2x - 5) + (-3x + 7)$$
$$= [2x + (-3x)] + [(-5) + 7]$$
$$= -x + 2$$

The distributive property is used to find the product of polynomials. For instance, to find the product of $(3x - 4)$ and $(2x^2 + 5x + 1)$, we treat $3x - 4$ as a *single* quantity and *distribute* it over the trinomial $2x^2 + 5x + 1$, as shown in Example 3.

Alternative to Example 3
a. Multiply: $(x^2 - 5x - 6)(2x - 5)$
■ $2x^3 - 15x^2 + 13x + 30$
b. Multiply: $(2x^3 + 3x - 4)(x + 5)$
■ $2x^4 + 10x^3 + 3x^2 + 11x - 20$

EXAMPLE 3 Multiply Polynomials

Simplify: $(3x - 4)(2x^2 + 5x + 1)$

Solution

$$(3x - 4)(2x^2 + 5x + 1)$$
$$= (3x - 4)(2x^2) + (3x - 4)(5x) + (3x - 4)(1)$$
$$= (3x)(2x^2) - 4(2x^2) + (3x)(5x) - 4(5x) + (3x)(1) - 4(1)$$
$$= 6x^3 - 8x^2 + 15x^2 - 20x + 3x - 4$$
$$= 6x^3 + 7x^2 - 17x - 4$$

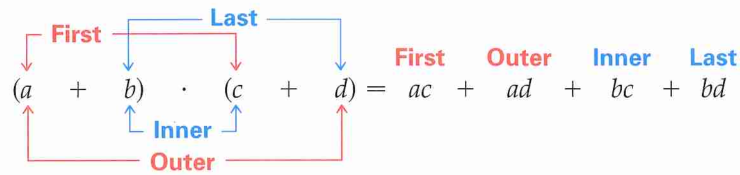
Try Exercise 32, page 40

In the following calculation, a vertical format has been used to find the product of $(x^2 + 6x - 7)$ and $(5x - 2)$. Note that like terms are arranged in the same vertical column.

ANSWER The additive inverse is $-3x^2 + 8x - 7$.

$$\begin{array}{r}
x^2 + 6x - 7 \\
5x - 2 \\
\hline
-\ 2x^2 - 12x + 14 = -2(x^2 + 6x - 7) \\
5x^3 + 30x^2 - 35x \phantom{{}+14} = 5x(x^2 + 6x - 7) \\
\hline
5x^3 + 28x^2 - 47x + 14
\end{array}$$

If the terms of the binomials $(a + b)$ and $(c + d)$ are labeled as shown below, then the product of the two binomials can be computed mentally by the **FOIL method.**



In the following illustration, we find the product of $(7x - 2)$ and $(5x + 4)$ by the FOIL method.

$$\begin{aligned}
& \qquad\quad \text{First} \qquad \text{Outer} \qquad \text{Inner} \qquad \text{Last} \\
(7x - 2)(5x + 4) &= (7x)(5x) + (7x)(4) + (-2)(5x) + (-2)(4) \\
&= 35x^2 + 28x - 10x - 8 \\
&= 35x^2 + 18x - 8
\end{aligned}$$

Certain products occur so frequently in algebra that they deserve special attention.

Special Product Formulas

Special Form	Formula(s)
(Sum)(Difference)	$(x + y)(x - y) = x^2 - y^2$
$(\text{Binomial})^2$	$(x + y)^2 = x^2 + 2xy + y^2$ $(x - y)^2 = x^2 - 2xy + y^2$

The variables x and y in these special product formulas can be replaced by other algebraic expressions, as shown in Example 4.

Alternative to Example 4

a. Multiply: $(2x^2 - 3)(2x^2 + 3)$

■ $4x^4 - 9$

b. Multiply: $(3x + 4y)^2$

■ $9x^2 + 24xy + 16y^2$

EXAMPLE 4 ⟫ Use the Special Product Formulas

Find each special product. **a.** $(7x + 10)(7x - 10)$ **b.** $(2y^2 + 11z)^2$

Solution

a. $(7x + 10)(7x - 10) = (7x)^2 - (10)^2 = 49x^2 - 100$

b. $(2y^2 + 11z)^2 = (2y^2)^2 + 2[(2y^2)(11z)] + (11z)^2 = 4y^4 + 44y^2z + 121z^2$

⟫ *Try Exercise 56, page 40*

Many application problems require you to *evaluate polynomials*. To **evaluate a polynomial,** substitute the given value(s) for the variable(s) and then perform the indicated operations using the Order of Operations Agreement.

Alternative to Example 5

a. Evaluate $3x^3 + 4x^2 - 6x - 7$ when $x = -2$.

■ **−3**

b. Evaluate $x^2 - x + 1$ when $x = \frac{1}{2}$.

■ $\frac{3}{4}$

EXAMPLE 5 Evaluate a Polynomial

Evaluate the polynomial $2x^3 - 6x^2 + 7$ for $x = -4$.

Solution

$2x^3 - 6x^2 + 7$

$2(-4)^3 - 6(-4)^2 + 7 = 2(-64) - 6(16) + 7$ • Substitute −4 for *x*. Evaluate the powers.

$= -128 - 96 + 7$ • Perform the multiplications.

$= -217$ • Perform the additions and subtractions.

Try Exercise 66, page 40

Applications of Polynomials

Alternative to Example 6

Exercise 72, page 41.

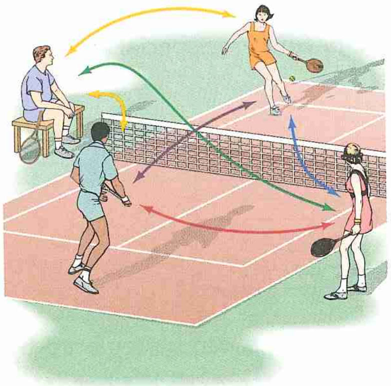


Figure P.13

Four tennis players can play a total of six singles matches.

EXAMPLE 6 Solve an Application

The number of singles tennis matches that can be played among n tennis players is given by the polynomial $\frac{1}{2}n^2 - \frac{1}{2}n$. Find the number of singles tennis matches that can be played among four tennis players.

Solution

$\frac{1}{2}n^2 - \frac{1}{2}n$

$\frac{1}{2}(4)^2 - \frac{1}{2}(4) = \frac{1}{2}(16) - \frac{1}{2}(4) = 8 - 2 = 6$ • Substitute 4 for *n*. Then simplify.

Therefore, four tennis players can play a total of six singles matches. See **Figure P.13.**

Try Exercise 76, page 41

Alternative to Example 7

Exercise 74, page 41.

EXAMPLE 7 Solve an Application

A scientist determines that the average time in seconds that it takes a particular computer to determine whether an n-digit natural number is prime or composite is given by

$$0.002n^2 + 0.002n + 0.009, \quad 20 \le n \le 40$$

The average time in seconds that it takes the computer to factor an n-digit number is given by

$$0.00032(1.7)^n, \quad 20 \le n \le 40$$

Math Matters

The procedure used by the computer to determine whether a number is prime or composite is a *polynomial time algorithm*, because the time required can be estimated using a polynomial. The procedure used to factor a number is an *exponential time algorithm*. In the field of *computational complexity*, it is important to distinguish between polynomial time algorithms and exponential time algorithms. Example 7 illustrates that the polynomial time algorithm can be run in about 2 seconds, whereas the exponential time algorithm requires about 44 minutes!

Estimate the average time it takes the computer to

a. determine whether a 30-digit number is prime or composite

b. factor a 30-digit number

Solution

a. $0.002n^2 + 0.002n + 0.009$

$0.002(30)^2 + 0.002(30) + 0.009 = 1.8 + 0.06 + 0.009 = 1.869 \approx 2$ seconds

b. $0.00032(1.7)^n$

$0.00032(1.7)^{30} \approx 0.00032(8{,}193{,}465.726)$

≈ 2600 seconds

Try Exercise 78, page 41

Topics for Discussion

1. Discuss the definition of the term *polynomial*. Give some examples of expressions that are polynomials and expressions that are not polynomials.
2. Suppose that P and Q are both polynomials of degree n. Discuss the degrees of $P + Q$, $P - Q$, PQ, $P + P$, and $P - P$.
3. Suppose that you evaluate a polynomial P of degree n for larger and larger values of x (for instance, when $x = 1, 2, 3, 4, \ldots$). Discuss whether the value of the polynomial would eventually (for very large values of x) continually increase, continually decrease, or fluctuate between increasing and decreasing.
4. Discuss the similarities and differences among monomials, binomials, trinomials, and polynomials.

—Suggested Assignment: Exercises 1–69, every other odd; 71–83, odd.

Exercise Set P.3

In Exercises 1 to 10, match the descriptions, labeled A, B, C,…, J, with the appropriate examples.

A. $x^3y + xy$ B. $7x^2 + 5x - 11$

C. $\frac{1}{2}x^2 + xy + y^2$ D. $4xy$

E. $8x^3 - 1$ F. $3 - 4x^2$

G. 8 H. $3x^5 - 4x^2 + 7x - 11$

I. $8x^4 - \sqrt{5}x^3 + 7$ J. 0

1. A monomial of degree 2 D
2. A binomial of degree 3 E
3. A polynomial of degree 5 H
4. A binomial with a leading coefficient of -4 F
5. A zero-degree polynomial G
6. A fourth-degree polynomial that has a third-degree term I
7. A trinomial with integer coefficients B
8. A trinomial in x and y C
9. A polynomial with no degree J
10. A fourth-degree binomial A

In Exercises 11 to 16, for each polynomial, determine its *a.* standard form, *b.* degree, *c.* coefficients, *d.* leading coefficient, and *e.* terms.

11. $2x + x^2 - 7$ a. $x^2 + 2x - 7$ b. 2 c. 1, 2, −7 d. 1 e. $x^2, 2x, -7$

12. $-3x^2 - 11 - 12x^4$ a. $-12x^4 - 3x^2 - 11$ b. 4 c. −12, −3, −11 d. −12 e. $-12x^4, -3x^2, -11$

13. $x^3 - 1$ a. $x^3 - 1$ b. 3 c. 1, −1 d. 1 e. $x^3, -1$

14. $4x^2 - 2x + 7$ a. $4x^2 - 2x + 7$ b. 2 c. 4, −2, 7 d. 4 e. $4x^2, -2x, 7$

15. $2x^4 + 3x^3 + 5 + 4x^2$ a. $2x^4 + 3x^3 + 4x^2 + 5$ b. 4 c. 2, 3, 4, 5 d. 2 e. $2x^4, 3x^3, 4x^2, 5$

16. $3x^2 - 5x^3 + 7x - 1$ a. $-5x^3 + 3x^2 + 7x - 1$ b. 3 c. −5, 3, 7, −1 d. −5 e. $-5x^3, 3x^2, 7x, -1$

In Exercises 17 to 22, determine the degree of the given polynomial.

17. $3xy^2 - 2xy + 7x$ 3

18. $x^3 + 3x^2y + 3xy^2 + y^3$ 3

19. $4x^2y^2 - 5x^3y^2 + 17xy^3$ 5

20. $-9x^5y + 10xy^4 - 11x^2y^2$ 6

21. xy 2

22. $5x^2y - y^4 + 6xy$ 4

In Exercises 23 to 34, perform the indicated operation and simplify if possible by combining like terms. Write the result in standard form.

23. $(3x^2 + 4x + 5) + (2x^2 + 7x - 2)$ $5x^2 + 11x + 3$

24. $(5y^2 - 7y + 3) + (2y^2 + 8y + 1)$ $7y^2 + y + 4$

25. $(4w^3 - 2w + 7) + (5w^3 + 8w^2 - 1)$ $9w^3 + 8w^2 - 2w + 6$

26. $(5x^4 - 3x^2 + 9) + (3x^3 - 2x^2 - 7x + 3)$ $5x^4 + 3x^3 - 5x^2 - 7x + 12$

27. $(r^2 - 2r - 5) - (3r^2 - 5r + 7)$ $-2r^2 + 3r - 12$

28. $(7s^2 - 4s + 11) - (-2s^2 + 11s - 9)$ $9s^2 - 15s + 20$

29. $(u^3 - 3u^2 - 4u + 8) - (u^3 - 2u + 4)$ $-3u^2 - 2u + 4$

30. $(5v^4 - 3v^2 + 9) - (6v^4 + 11v^2 - 10)$ $-v^4 - 14v^2 + 19$

31. $(4x - 5)(2x^2 + 7x - 8)$ $8x^3 + 18x^2 - 67x + 40$

32. $(5x - 7)(3x^2 - 8x - 5)$ $15x^3 - 61x^2 + 31x + 35$

33. $(3x^2 - 2x + 5)(2x^2 - 5x + 2)$ $6x^4 - 19x^3 + 26x^2 - 29x + 10$

34. $(2y^3 - 3y + 4)(2y^2 - 5y + 7)$ $4y^5 - 10y^4 + 8y^3 + 23y^2 - 41y + 28$

In Exercises 35 to 48, use the FOIL method to find the indicated product.

35. $(2x + 4)(5x + 1)$ $10x^2 + 22x + 4$

36. $(5x - 3)(2x + 7)$ $10x^2 + 29x - 21$

37. $(y + 2)(y + 1)$ $y^2 + 3y + 2$

38. $(y + 5)(y + 3)$ $y^2 + 8y + 15$

39. $(4z - 3)(z - 4)$ $4z^2 - 19z + 12$

40. $(5z - 6)(z - 1)$ $5z^2 - 11z + 6$

41. $(a + 6)(a - 3)$ $a^2 + 3a - 18$

42. $(a - 10)(a + 4)$ $a^2 - 6a - 40$

43. $(5x - 11y)(2x - 7y)$ $10x^2 - 57xy + 77y^2$

44. $(3a - 5b)(4a - 7b)$ $12a^2 - 41ab + 35b^2$

45. $(9x + 5y)(2x + 5y)$ $18x^2 + 55xy + 25y^2$

46. $(3x - 7z)(5x - 7z)$ $15x^2 - 56xz + 49z^2$

47. $(3p + 5q)(2p - 7q)$ $6p^2 - 11pq - 35q^2$

48. $(2r - 11s)(5r + 8s)$ $10r^2 - 39rs - 88s^2$

In Exercises 49 to 54, perform the indicated operation(s) and simplify.

49. $(4d - 1)^2 - (2d - 3)^2$ $12d^2 + 4d - 8$

50. $(5c - 8)^2 - (2c - 5)^2$ $21c^2 - 60c + 39$

51. $(r + s)(r^2 - rs + s^2)$ $r^3 + s^3$

52. $(r - s)(r^2 + rs + s^2)$ $r^3 - s^3$

53. $(3c - 2)(4c + 1)(5c - 2)$ $60c^3 - 49c^2 + 4$

54. $(4d - 5)(2d - 1)(3d - 4)$ $24d^3 - 74d^2 + 71d - 20$

In Exercises 55 to 62, use the special product formulas to perform the indicated operation.

55. $(3x + 5)(3x - 5)$ $9x^2 - 25$

56. $(4x^2 - 3y)(4x^2 + 3y)$ $16x^4 - 9y^2$

57. $(3x^2 - y)^2$ $9x^4 - 6x^2y + y^2$

58. $(6x + 7y)^2$ $36x^2 + 84xy + 49y^2$

59. $(4w + z)^2$ $16w^2 + 8wz + z^2$

60. $(3x - 5y^2)^2$ $9x^2 - 30xy^2 + 25y^4$

61. $[(x + 5) + y][(x + 5) - y]$ $x^2 + 10x + 25 - y^2$

62. $[(x - 2y) + 7][(x - 2y) - 7]$ $x^2 - 4xy + 4y^2 - 49$

In Exercises 63 to 70, evaluate the given polynomial for the indicated value of the variable.

63. $x^2 + 7x - 1$, for $x = 3$ 29

64. $x^2 - 8x + 2$, for $x = 4$ −14

65. $-x^2 + 5x - 3$, for $x = -2$ −17

66. $-x^2 - 5x + 4$, for $x = -5$ 4

67. $3x^3 - 2x^2 - x + 3$, for $x = -1$ −1

68. $5x^3 - x^2 + 5x - 3$, for $x = -1$ −14

69. $1 - x^5$, for $x = -2$ 33

70. $1 - x^3 - x^5$, for $x = 2$ −39

71. RECREATION The air resistance (in pounds) on a cyclist riding a bicycle in an upright position can be given by $0.016v^2$, where v is the speed of the cyclist in miles per hour. Find the air resistance on a cyclist when

a. $v = 10$ mph 1.6 lb **b.** $v = 15$ mph 3.6 lb

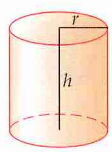
72. HIGHWAY ENGINEERING On an expressway, the recommended *safe distance* between cars in feet is given by $0.015v^2 + v + 10$, where v is the speed of the car in miles per hour. Find the safe distance when

a. $v = 30$ mph 53.5 ft **b.** $v = 55$ mph 110.375 ft

73. GEOMETRY The volume of a right circular cylinder (as shown below) is given by $\pi r^2 h$, where r is the radius of the base and h is the height of the cylinder. Find the volume when

a. $r = 3$ inches, $h = 8$ inches
72π in.^{3}

b. $r = 5$ cm, $h = 12$ cm
300π cm^3



74. AUTOMOTIVE ENGINEERING The fuel efficiency (in miles per gallon of gas) of a car is given by the expression $-0.02v^2 + 1.5v + 2$, where v is the speed of the car in miles per hour. Find the fuel efficiency when

a. $v = 45$ mph
29 mi/gal

b. $v = 60$ mph
20 mi/gal

75. PSYCHOLOGY Based on data from one experiment, the reaction time, in hundredths of a second, of a person to visual stimulus varies according to age and is given by $0.005x^2 - 0.32x + 12$, where x is the age of the person. Find the reaction time to the stimulus for a person who is

a. $x = 20$ years old
0.076 s

b. $x = 50$ years old
0.085 s

76. COMMITTEE MEMBERSHIP The number of committees consisting of exactly 3 people that can be formed from a group of n people is given by the polynomial

$$\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{1}{3}n$$

Find the number of committees consisting of exactly 3 people that can be formed from a group of 21 people.
1330 committees

77. CHESS MATCHES Find the number of chess matches that can be played between the members of a group of 150 people. Use the formula from Example 6. 11,175 matches

78. COMPUTER SCIENCE A computer scientist determines that the time in seconds it takes a particular computer to calculate n digits of π is given by the polynomial

$$4.3 \times 10^{-6}n^2 - 2.1 \times 10^{-4}n$$

where $1000 \le n \le 10{,}000$. Estimate the time it takes the computer to calculate π to

a. 1000 digits
4.09 s

b. 5000 digits
106.45 s

c. 10,000 digits
427.9 s

79. COMPUTER SCIENCE If n is a positive integer, then $n!$, which is read "n factorial," is given by

$$n(n-1)(n-2)\cdots 2 \cdot 1$$

For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. A computer scientist determines that each time a program is run on a particular computer, the time in seconds required to compute $n!$ is given by the polynomial

$$1.9 \times 10^{-6}n^2 - 3.9 \times 10^{-3}n$$

where $1000 \le n \le 10{,}000$. Using this polynomial, estimate the time it takes this computer to calculate 4000! and 8000!.
14.8 s; 90.4 s

80. AIR VELOCITY OF A COUGH The velocity, in meters per second, of the air that is expelled during a cough is given by velocity $= 6r^2 - 10r^3$, where r is the radius of the trachea in centimeters.

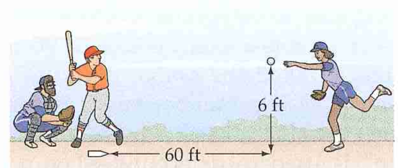
a. Find the velocity as a polynomial in standard form.
$-10r^3 + 6r^2$ m/s

b. Find the velocity of the air in a cough when the radius of the trachea is 0.35 cm. Round to the nearest hundredth. 0.31 m/s

81. SPORTS The height, in feet, of a baseball released by a pitcher t seconds after it is released is given by (ignoring air resistance)

$$\text{Height} = -16t^2 + 4.7881t + 6$$

For the pitch to be a strike, it must be at least 2 feet high and no more than 5 feet high when it crosses home plate. If it takes 0.5 second for the ball to reach home plate, will the ball be high enough to be a strike?
Yes. The ball is approximately 4.4 ft high when it crosses home plate.



Not to scale

82. MEDICINE The temperature, in degrees Fahrenheit, of a patient after receiving a certain medication is given by

$$\text{Temperature} = 0.0002t^3 - 0.0114t^2 + 0.0158t + 104$$

where t is the number of minutes after receiving the medication.

a. What was the patient's temperature just before the medication was given? 104°F

b. What was the patient's temperature 25 minutes after the medication was given? ≈100.4°F

83. PRIME FACTORIZATION Find n given that

$$n! = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$$ 15

»»» Connecting Concepts

The following special product formulas can be used to find the cube of a binomial.

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

In Exercises 84 to 89, make use of the given special product formulas to find the indicated products.

84. $(a + b)^3$ $a^3 + 3a^2b + 3ab^2 + b^3$

85. $(a - b)^3$ $a^3 - 3a^2b + 3ab^2 - b^3$

86. $(x - 1)^3$ $x^3 - 3x^2 + 3x - 1$

87. $(y + 2)^3$ $y^3 + 6y^2 + 12y + 8$

88. $(2x - 3y)^3$ $8x^3 - 36x^2y + 54xy^2 - 27y^3$

89. $(3x + 5y)^3$ $27x^3 + 135x^2y + 225xy^2 + 125y^3$

»»» Projects

1. **ODD NUMBERS** Every odd number can be written in the form $2n - 1$, and every even number can be expressed as $2n$, where n is a natural number. Explain, by writing a few paragraphs and giving the supporting mathematics, why the product of two odd numbers is an odd number, the product of two even numbers is an even number, and the product of an even number and an odd number is an even number.

2. **PRIME NUMBERS** Fermat's Little Theorem states, "If n is a prime number and a is *any* natural number, then $a^n - a$ is divisible by n." For instance, for $n = 11$ and $a = 14$, $\dfrac{14^{11} - 14}{11} = 368{,}142{,}288{,}150$. The important aspect of this theorem is that no matter what natural number is chosen for a, $a^{11} - a$ is evenly divisible by 11.

Being able to determine whether a number is prime plays a central role in many computer security systems. A restatement (called the *contrapositive*) of Fermat's Little Theorem, "If n is a number and a is some number for which $a^n - a$ is *not* divisible by n, then n is *not* a prime number," is used to determine when a number is *not* prime. For example, if $n = 14$, then $\dfrac{2^{14} - 2}{14} = \dfrac{8191}{7}$, and thus there is some number a ($a = 2$) for which $a^{14} - a$ is not evenly divisible by 14. Therefore, 14 is not prime.

a. Explain the meaning of the *contrapositive* (used above) of a theorem. Use your explanation to write the contrapositive of "If two triangles are congruent, then they are similar."

b. $7^{14} - 7$ is divisible by 14. Explain why this does not contradict the fact that 14 is not prime.

c. Explain the meaning of the *converse* of a theorem. State the converse of Fermat's Little Theorem.

d. The number 561 has the property that $a^{561} - a$ is divisible by 561 for all natural numbers a. Can you use Fermat's Little Theorem to conclude that 561 is a prime number? Explain.

e. Suppose that $a^n - a$ is divisible by n for all values of a. Can you conclude that n is a prime number? Explain your answer.

f. Find a definition of a Carmichael number. What do Carmichael numbers have to do with the information in **e.**?

Section P.4

- Greatest Common Factor
- Factoring Trinomials
- Special Factoring
- Factor by Grouping
- General Factoring

Factoring

PREPARE FOR THIS SECTION

Prepare for this section by completing the following exercises. The answers can be found on page A2.

PS1. Simplify: $\frac{6x^3}{2x}$ [P.2] $3x^2$

PS2. Simplify: $(-12x^4)3x^2$ [P.2] $-36x^6$

PS3. Express x^6 as a power of **a.** x^2 and **b.** x^3. [P.2] **a.** $(x^2)^3$ **b.** $(x^3)^2$

In Exercises PS4 to PS6, replace the question mark to make a true statement.

PS4. $6a^3b^4 \cdot ? = 18a^3b^7$ [P.2] $3b^3$

PS5. $-3(5a - ?) = -15a + 21$ [P.1] 7

PS6. $2x(3x - ?) = 6x^2 - 2x$ [P.1] 1

Writing a polynomial as a product of polynomials of lower degree is called **factoring.** Factoring is an important procedure that is often used to simplify fractional expressions and to solve equations.

In this section we consider only the factorization of polynomials that have integer coefficients. Also, we are concerned only with **factoring over the integers.** That is, we search only for polynomial factors that have integer coefficients.

Greatest Common Factor

The first step in the factorization of any polynomial is to use the distributive property to factor out the **greatest common factor (GCF)** of the terms of the polynomial. Given two or more exponential expressions with the same prime number base or the same variable base, the GCF is the exponential expression with the smallest exponent. For example,

$$2^3 \text{ is the GCF of } 2^3, 2^5, \text{ and } 2^8 \quad \text{and} \quad a \text{ is the GCF of } a^4 \text{ and } a$$

The GCF of two or more monomials is the product of the GCFs of all the *common* bases. For example, to find the GCF of $27a^3b^4$ and $18b^3c$, factor the coefficients into prime factors and then write each common base with its smallest exponent.

$$27a^3b^4 = 3^3 \cdot a^3 \cdot b^4 \qquad 18b^3c = 2 \cdot 3^2 \cdot b^3 \cdot c$$

The only common bases are 3 and b. The product of these common bases with their smallest exponents is 3^2b^3. The GCF of $27a^3b^4$ and $18b^3c$ is $9b^3$.

The expressions $3x(2x + 5)$ and $4(2x + 5)$ have a common *binomial* factor, which is $2x + 5$. Thus the GCF of $3x(2x + 5)$ and $4(2x + 5)$ is $2x + 5$.

Alternative to Example 1

Factor out the GCF.

a. $-6x^2y^2 + 3xy^2$

■ $\mathbf{-3xy^2(2x - 1)}$

b. $2x(3x + 1) - (3x + 1)$

■ $\mathbf{(3x + 1)(2x - 1)}$

EXAMPLE 1 ⟫ Factor Out the Greatest Common Factor

Factor out the GCF.

a. $10x^3 + 6x$ **b.** $15x^{2n} + 9x^{n+1} - 3x^n$ (where n is a positive integer)

c. $(m + 5)(x + 3) + (m + 5)(x - 10)$

Continued ▶

Solution

a. $10x^3 + 6x = (2x)(5x^2) + (2x)(3)$ • The GCF is $2x$.

$= 2x(5x^2 + 3)$ • Factor out the GCF.

b. $15x^{2n} + 9x^{n+1} - 3x^n$

$= (3x^n)(5x^n) + (3x^n)(3x) - (3x^n)(1)$ • The GCF is $3x^n$.

$= 3x^n(5x^n + 3x - 1)$ • Factor out the GCF.

c. $(m + 5)(x + 3) + (m + 5)(x - 10)$ • Use the distributive property to factor out $(m + 5)$.

$= (m + 5)[(x + 3) + (x - 10)]$

$= (m + 5)(2x - 7)$ • Simplify.

Try Exercise 6, page 52

Factoring Trinomials

Some trinomials of the form $x^2 + bx + c$ can be factored by a trial procedure. This method makes use of the FOIL method in reverse. For example, consider the following products:

$$(x + 3)(x + 5) = x^2 + 5x + 3x + (3)(5) \quad = x^2 + 8x + 15$$
$$(x - 2)(x - 7) = x^2 - 7x - 2x + (-2)(-7) = x^2 - 9x + 14$$
$$(x + 4)(x - 9) = x^2 - 9x + 4x + (4)(-9) \quad = x^2 - 5x - 36$$

The coefficient of x is the sum of the constant terms of the binomials.

The constant term of the trinomial is the product of the constant terms of the binomials.

QUESTION Is $(x - 2)(x + 7)$ the correct factorization of $x^2 - 5x - 14$?

Points to Remember to Factor $x^2 + bx + c$

1. The constant term c of the trinomial is the product of the constant terms of the binomials.
2. The coefficient b in the trinomial is the sum of the constant terms of the binomials.
3. If the constant term c of the trinomial is positive, the constant terms of the binomials have the same sign as the coefficient b in the trinomial.
4. If the constant term c of the trinomial is negative, the constant terms of the binomials have opposite signs.

ANSWER No. $(x - 2)(x + 7) = x^2 + 5x - 14$.

Alternative to Example 2

a. Factor: $x^2 - 5x - 6$

■ $(x - 6)(x + 1)$

b. Factor: $x^2 + 6x - 27$

■ $(x + 9)(x - 3)$

EXAMPLE 2 Factor a Trinomial of the Form $x^2 + bx + c$

Factor: $x^2 + 7x - 18$

Solution

We must find two binomials whose first terms have a product of x^2 and whose last terms have a product of -18; also, the sum of the product of the outer terms and the product of the inner terms must be $7x$. Begin by listing the possible integer factorizations of -18 and the sums of those factors.

Factors of −18	Sum of the Factors
$1 \cdot (-18)$	$1 + (-18) = -17$
$(-1) \cdot 18$	$(-1) + 18 = 17$
$2 \cdot (-9)$	$2 + (-9) = -7$
$(-2) \cdot 9$	$(-2) + 9 = 7$

• Stop. This is the desired sum.

Thus -2 and 9 are the numbers whose sum is 7 and whose product is -18. Therefore,

$$x^2 + 7x - 18 = (x - 2)(x + 9)$$

The FOIL method can be used to verify that the factorization is correct.

Try Exercise 12, page 52

TO REVIEW

FOIL

See page 37.

The trial method sometimes can be used to factor trinomials of the form $ax^2 + bx + c$, which do not have a leading coefficient of 1. We use the factors of a and c to form trial binomial factors. Factoring trinomials of this type may require testing many factors. To reduce the number of trial factors, make use of the following points.

Points to Remember to Factor $ax^2 + bx + c$, $a > 0$

1. If the constant term of the trinomial is positive, the constant terms of the binomials have the same sign as the coefficient b in the trinomial.
2. If the constant term of the trinomial is negative, the constant terms of the binomials have opposite signs.
3. If the terms of the trinomial do not have a common factor, then neither binomial will have a common factor.

Alternative to Example 3

a. Factor: $6x^2 + 17x - 10$

■ $(2x - 1)(3x + 10)$

b. Factor: $4x^2 - 17x - 21$

■ $(4x - 21)(x + 1)$

EXAMPLE 3 Factor a Trinomial of the Form $ax^2 + bx + c$

Factor: $6x^2 - 11x + 4$

Solution

Because the constant term of the trinomial is positive and the coefficient of the x term is negative, the constant terms of the binomials will both be negative. This time we find factors of the first term as well as factors of the constant term.

Factors of $6x^2$	Factors of 4 (both negative)
$x, 6x$	$-1, -4$
$2x, 3x$	$-2, -2$

Use these factors to write trial factors. Use the FOIL method to see whether any of the trial factors produce the correct middle term. If the terms of a trinomial do not have a common factor, then a binomial factor cannot have a common factor (point 3). Such trial factors need not be checked.

Trial Factors	Middle Term	
$(x - 1)(6x - 4)$	Common factor	• $6x$ and 4 have a common factor.
$(x - 4)(6x - 1)$	$-1x - 24x = -25x$	
$(x - 2)(6x - 2)$	Common factor	• $6x$ and 2 have a common factor.
$(2x - 1)(3x - 4)$	$-8x - 3x = -11x$	• This is the correct middle term.

Thus $6x^2 - 11x + 4 = (2x - 1)(3x - 4)$.

Try Exercise 16, page 52

Sometimes it is impossible to factor a polynomial into the product of two polynomials having integer coefficients. Such polynomials are said to be **nonfactorable over the integers.** For example, $x^2 + 3x + 7$ is nonfactorable over the integers because there are no integers whose product is 7 and whose sum or difference is 3.

If you have difficulty factoring a trinomial, you may wish to use the following theorem. It will indicate whether the trinomial is factorable over the integers.

Factorization Theorem

The trinomial $ax^2 + bx + c$, with integer coefficients a, b, and c, can be factored as the product of two binomials with integer coefficients if and only if $b^2 - 4ac$ is a perfect square.

Alternative to Example 4

a. Determine whether $2x^2 + x - 3$ is factorable over the integers.

■ **Yes**

b. Determine whether $4x^2 + 2x + 9$ is factorable over the integers.

■ **No**

EXAMPLE 4 Apply the Factorization Theorem

Determine whether each trinomial is factorable over the integers.

a. $4x^2 + 8x - 7$

b. $6x^2 - 5x - 4$

Solution

a. The coefficients of $4x^2 + 8x - 7$ are $a = 4$, $b = 8$, and $c = -7$. Applying the factorization theorem yields

$$b^2 - 4ac = 8^2 - 4(4)(-7) = 176$$

Because 176 is not a perfect square, the trinomial is nonfactorable over the integers.

b. The coefficients of $6x^2 - 5x - 4$ are $a = 6$, $b = -5$, and $c = -4$. Thus

$$b^2 - 4ac = (-5)^2 - 4(6)(-4) = 121$$

Because 121 is a perfect square, the trinomial is factorable over the integers. Using the methods we have developed, we find

$$6x^2 - 5x - 4 = (3x - 4)(2x + 1)$$

Try Exercise 24, page 52

Certain trinomials can be expressed as quadratic trinomials by making suitable variable substitutions. A trinomial is **quadratic in form** if it can be written as

$$au^2 + bu + c$$

If we let $x^2 = u$, the trinomial $x^4 + 5x^2 + 6$ can be written as shown at the right.

$$x^4 + 5x^2 + 6 = (x^2)^2 + 5(x^2) + 6$$

The trinomial is quadratic in form.

$$= u^2 + 5u + 6$$

If we let $xy = u$, the trinomial $2x^2y^2 + 3xy - 9$ can be written as shown at the right.

$$2x^2y^2 + 3xy - 9 = 2(xy)^2 + 3(xy) - 9$$

The trinomial is quadratic in form.

$$= 2u^2 + 3u - 9$$

When a trinomial that is quadratic in form is factored, the variable part of the first term in each binomial factor will be u. For example, because $x^4 + 5x^2 + 6$ is quadratic in form when $x^2 = u$, the first term in each binomial factor will be x^2.

$$\begin{aligned} x^4 + 5x^2 + 6 &= (x^2)^2 + 5(x^2) + 6 \\ &= (x^2 + 2)(x^2 + 3) \end{aligned}$$

The trinomial $x^2y^2 - 2xy - 15$ is quadratic in form when $xy = u$. The first term in each binomial factor will be xy.

$$\begin{aligned} x^2y^2 - 2xy - 15 &= (xy)^2 - 2(xy) - 15 \\ &= (xy + 3)(xy - 5) \end{aligned}$$

Alternative to Example 5
Factor: $3x^4 + 4x^2 - 4$
■ $(3x^2 - 2)(x^2 + 2)$

EXAMPLE 5 Factor a Trinomial in Quadratic Form

Factor. **a.** $6x^2y^2 - xy - 12$ **b.** $2x^4 + 5x^2 - 12$

Solution

a. $6x^2y^2 - xy - 12$
$= (3xy + 4)(2xy - 3)$

• The trinomial is quadratic in form when $xy = u$.

b. $2x^4 + 5x^2 - 12$
$= (x^2 + 4)(2x^2 - 3)$

• The trinomial is quadratic in form when $x^2 = u$.

Try Exercise 36, page 52

Special Factoring

The product of a term and itself is called a **perfect square.** The exponents on variables of perfect squares are always even numbers. The **square root of a perfect square** is one of the two equal factors of the perfect square. To find the square root of a perfect square variable term, divide the exponent by 2. For the examples below, assume the variables represent positive numbers.

Term		Perfect Square	Square Root
7	$7 \cdot 7 =$	49	$\sqrt{49} = 7$
y	$y \cdot y =$	y^2	$\sqrt{y^2} = y$
$2x^3$	$2x^3 \cdot 2x^3 =$	$4x^6$	$\sqrt{4x^6} = 2x^3$
x^n	$x^n \cdot x^n =$	x^{2n}	$\sqrt{x^{2n}} = x^n$

take note

The **sum** of two squares does not factor over the integers. For instance, $49x^2 + 144$ does not factor over the integers.

The factors of the difference of two perfect squares are the sum and difference of the square roots of the perfect squares.

Factors of the Difference of Two Perfect Squares

$$a^2 - b^2 = (a + b)(a - b)$$

Alternative to Example 6
a. Factor: $2x^2 - 18$
■ $2(x + 3)(x - 3)$
b. Factor: $x^2y^2 - 81$
■ $(xy + 9)(xy - 9)$

EXAMPLE 6 Factor the Difference of Squares

Factor: $49x^2 - 144$

Solution

$49x^2 - 144 = (7x)^2 - (12)^2$

• Recognize the difference-of-squares form.

$= (7x + 12)(7x - 12)$

• The binomial factors are the sum and difference of the square roots of the squares.

Try Exercise 40, page 52

A **perfect-square trinomial** is a trinomial that is the square of a binomial. For example, $x^2 + 6x + 9$ is a perfect-square trinomial because

$$(x + 3)^2 = x^2 + 6x + 9$$

Every perfect-square trinomial can be factored by the trial method, but it generally is faster to factor perfect-square trinomials by using the following factoring formulas.

Factors of a Perfect-Square Trinomial

$$a^2 + 2ab + b^2 = (a + b)^2$$
$$a^2 - 2ab + b^2 = (a - b)^2$$

Alternative to Example 7

Factor: $12x^2 + 36x + 27$

■ $3(2x + 3)^2$

EXAMPLE 7 Factor a Perfect-Square Trinomial

Factor: $16m^2 - 40mn + 25n^2$

(4m − 5n)²

Solution

Because $16m^2 = (4m)^2$ and $25n^2 = (5n)^2$, try factoring $16m^2 - 40mn + 25n^2$ as the square of a binomial.

$$16m^2 - 40mn + 25n^2 \stackrel{?}{=} (4m - 5n)^2$$

Check:

$$\begin{aligned}(4m - 5n)^2 &= (4m - 5n)(4m - 5n)\\ &= 16m^2 - 20mn - 20mn + 25n^2\\ &= 16m^2 - 40mn + 25n^2\end{aligned}$$

The factorization checks. Therefore, $16m^2 - 40mn + 25n^2 = (4m - 5n)^2$.

Try Exercise 50, page 53

take note

It is important to check the proposed factorization. For instance, consider $x^2 + 13x + 36$. Because x^2 is the square of x and 36 is the square of 6, it is tempting to factor, using the perfect-square trinomial formulas, as $x^2 + 13x + 36 \stackrel{?}{=} (x + 6)^2$. Note, however, that $(x + 6)^2 = x^2 + 12x + 36$, which is not the original trinomial. The correct factorization is $x^2 + 13x + 36 = (x + 4)(x + 9)$.

The product of the same three terms is called a **perfect cube.** The exponents on variables of perfect cubes are always divisible by 3. The **cube root of a perfect cube** is one of the three equal factors of the perfect cube. To find the cube root of a perfect cube variable term, divide the exponent by 3.

Term		Perfect Cube	Cube Root
5	$5 \cdot 5 \cdot 5 =$	125	$\sqrt[3]{125} = 5$
z	$z \cdot z \cdot z =$	z^3	$\sqrt[3]{z^3} = z$
$3x^2$	$3x^2 \cdot 3x^2 \cdot 3x^2 =$	$27x^6$	$\sqrt[3]{27x^6} = 3x^2$
x^n	$x^n \cdot x^n \cdot x^n =$	x^{3n}	$\sqrt[3]{x^{3n}} = x^n$

The following factoring formulas are used to factor the sum or difference of two perfect cubes.

Factors of the Sum or Difference of Two Perfect Cubes

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

take note

Note the pattern of the signs when factoring the sum or difference of two perfect cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

Same sign (the sign in $a^3 + b^3$ and in $(a + b)$); Opposite signs (the sign in $a^3 + b^3$ and in $-ab$)

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Same sign (the sign in $a^3 - b^3$ and in $(a - b)$); Opposite signs (the sign in $a^3 - b^3$ and in $+ab$)

Alternative to Example 8

a. Factor: $27 - y^3$

■ $(3 - y)(9 + 3y + y^2)$

b. Factor: $x^4 + 8x$

■ $x(x + 2)(x^2 - 2x + 4)$

EXAMPLE 8 Factor the Sum or Difference of Cubes

Factor. **a.** $8a^3 + b^3$ **b.** $a^3 - 64$

Solution

a. $8a^3 + b^3 = (2a)^3 + b^3$ • Recognize the sum-of-cubes form.

$= (2a + b)(4a^2 - 2ab + b^2)$ • Factor.

b. $a^3 - 64 = a^3 - 4^3$ • Recognize the difference-of-cubes form.

$= (a - 4)(a^2 + 4a + 16)$ • Factor.

Try Exercise 56, page 53

Factor by Grouping

Some polynomials can be **factored by grouping.** Pairs of terms that have a common factor are first grouped together. The process makes repeated use of the distributive property, as shown in the following factorization of $6y^3 - 21y^2 - 4y + 14$.

take note

$-a + b = -(a - b)$. Thus, $-4y + 14 = -(4y - 14)$.

$6y^3 - 21y^2 - 4y + 14$

$= (6y^3 - 21y^2) - (4y - 14)$ • Group the first two terms and the last two terms.

$= 3y^2(2y - 7) - 2(2y - 7)$ • Factor out the GCF from each of the groups.

$= (2y - 7)(3y^2 - 2)$ • Factor out the common binomial factor.

When you factor by grouping, some experimentation may be necessary to find a grouping that fits the form of one of the special factoring formulas.

Alternative to Example 9

a. Factor: $2ax + 4bx - 3ay - 6by$

■ $(a + 2b)(2x - 3y)$

b. Factor: $xy - 3x - 4y + 12$

■ $(x - 4)(y - 3)$

EXAMPLE 9 Factor by Grouping

Factor by grouping. **a.** $a^2 + 10ab + 25b^2 - c^2$ **b.** $p^2 + p - q - q^2$

Solution

a. $a^2 + 10ab + 25b^2 - c^2$

$= (a^2 + 10ab + 25b^2) - c^2$ • Group the terms of the perfect-square trinomial.

$= (a + 5b)^2 - c^2$ • Factor the trinomial.

$= [(a + 5b) + c][(a + 5b) - c]$ • Factor the difference of squares.

$= (a + 5b + c)(a + 5b - c)$ • Simplify.

b. $p^2 + p - q - q^2$

$= p^2 - q^2 + p - q$ • Rearrange the terms.

$= (p^2 - q^2) + (p - q)$ • Regroup.

$= (p + q)(p - q) + (p - q)$ • Factor the difference of squares.

$= (p - q)(p + q + 1)$ • Factor out the common factor $(p - q)$.

Try Exercise 66, page 53

General Factoring

Here is a general factoring strategy for polynomials:

General Factoring Strategy

1. Factor out the GCF of all terms.
2. Try to factor a binomial as
   - **a.** the difference of two squares
   - **b.** the sum or difference of two cubes
3. Try to factor a trinomial
   - **a.** as a perfect-square trinomial
   - **b.** using the trial method
4. Try to factor a polynomial with more than three terms by grouping.
5. After each factorization, examine the new factors to see whether they can be factored.

Alternative to Example 10

a. Factor: $x^3 - 2x^2 - x + 2$

■ $(x - 2)(x + 1)(x - 1)$

b. Factor: $4x^2 + 4x + 1 - y^2$

■ $(2x + 1 - y)(2x + 1 + y)$

EXAMPLE 10 Factor Using the General Factoring Strategy

Completely factor: $x^6 + 7x^3 - 8$

Solution

Factor $x^6 + 7x^3 - 8$ as the product of two binomials.

$$x^6 + 7x^3 - 8 = (x^3 + 8)(x^3 - 1)$$

Now factor $x^3 + 8$, which is the sum of two cubes, and factor $x^3 - 1$, which is the difference of two cubes.

$$x^6 + 7x^3 - 8 = (x + 2)(x^2 - 2x + 4)(x - 1)(x^2 + x + 1)$$

Try Exercise 72, page 53

Topics for Discussion

1. Discuss the meaning of the phrase *nonfactorable over the integers.*
2. You know that if $ab = 0$, then $a = 0$ or $b = 0$. Suppose a polynomial is written in factored form and then set equal to zero. For instance, suppose

   $$x^2 - 2x - 15 = (x - 5)(x + 3) = 0$$

   Discuss what implications this has for the values of x. Do not answer this question only for the polynomial above, but also for any polynomial written as a product of linear factors and then set equal to zero.
3. Let P be a polynomial of degree n. Discuss the number of possible distinct linear polynomials that can be factors of P.

4. A method of evaluating polynomials, sometimes called Horner's method, involves factoring a polynomial in a certain manner. For instance,

$$4x^3 - 2x^2 + 5x - 3 = [(4x - 2)x + 5]x - 3$$
$$5x^4 - 2x^3 + 4x^2 + x - 6 = \{[(5x - 2)x + 4]x + 1\}x - 6$$

To evaluate the polynomial, the factored form is evaluated. Discuss the advantages and disadvantages of using this method to evaluate a polynomial.

5. If n is a natural number, $n! = n(n - 1)(n - 2) \cdots 3 \cdot 2 \cdot 1$. Explain why none of the following consecutive integers is a prime number.

$$5! + 2 \qquad 5! + 3 \qquad 5! + 4 \qquad 5! + 5$$

How many numbers are in the following list of consecutive integers? How many of those numbers are prime numbers?

$$k! + 2,\quad k! + 3,\quad k! + 4,\quad k! + 5,\quad \ldots,\quad k! + k$$

Explain why this result means that there are arbitrarily long sequences of consecutive natural numbers that do not contain a prime number.

—Suggested Assignment: Exercises 1–97, every other odd.

Exercise Set P.4

In Exercises 1 to 8, factor out the GCF from each polynomial.

1. $5x + 20$
   $5(x + 4)$
2. $8x^2 + 12x - 40$
   $4(2x^2 + 3x - 10)$
3. $-15x^2 - 12x$
   $-3x(5x + 4)$
4. $-6y^2 - 54y$
   $-6y(y + 9)$
5. $10x^2y + 6xy - 14xy^2$
   $2xy(5x + 3 - 7y)$
6. $6a^3b^2 - 12a^2b + 72ab^3$
   $6ab(a^2b - 2a + 12b^2)$
7. $(x - 3)(a + b) + (x - 3)(a + 2b)$
   $(x - 3)(2a + 3b)$
8. $(x - 4)(2a - b) + (x + 4)(2a - b)$
   $(2a - b)(2x)$

In Exercises 9 to 22, factor each trinomial over the integers.

9. $x^2 + 7x + 12$
   $(x + 3)(x + 4)$
10. $x^2 + 9x + 20$
   $(x + 4)(x + 5)$
11. $a^2 - 10a - 24$
   $(a - 12)(a + 2)$
12. $b^2 + 12b - 28$
   $(b + 14)(b - 2)$
13. $6x^2 + 25x + 4$
   $(6x + 1)(x + 4)$
14. $8a^2 - 26a + 15$
   $(4a - 3)(2a - 5)$
15. $51x^2 - 5x - 4$
   $(17x + 4)(3x - 1)$
16. $57y^2 + y - 6$
   $(19y - 6)(3y + 1)$
17. $6x^2 + xy - 40y^2$
   $(3x + 8y)(2x - 5y)$
18. $8x^2 + 10xy - 25y^2$
   $(4x - 5y)(2x + 5y)$
19. $x^4 + 6x^2 + 5$
   $(x^2 + 5)(x^2 + 1)$
20. $x^4 + 11x^2 + 18$
   $(x^2 + 9)(x^2 + 2)$
21. $6x^4 + 23x^2 + 15$
   $(6x^2 + 5)(x^2 + 3)$
22. $9x^4 + 10x^2 + 1$
   $(9x^2 + 1)(x^2 + 1)$

In Exercises 23 to 28, use the factorization theorem to determine whether each trinomial is factorable over the integers.

23. $8x^2 + 26x + 15$
   Factorable over the integers
24. $16x^2 + 8x - 35$
   Factorable over the integers
25. $4x^2 - 5x + 6$
   Not factorable over the integers
26. $6x^2 + 8x - 3$
   Not factorable over the integers
27. $6x^2 - 14x + 5$
   Not factorable over the integers
28. $10x^2 - 4x - 5$
   Not factorable over the integers

In Exercises 29 to 36, factor over the integers.

29. $x^4 - x^2 - 6$
   $(x^2 - 3)(x^2 + 2)$
30. $x^4 + 3x^2 + 2$
   $(x^2 + 1)(x^2 + 2)$
31. $x^2y^2 - 2xy - 8$
   $(xy - 4)(xy + 2)$
32. $2x^2y^2 + xy - 1$
   $(2xy - 1)(xy + 1)$
33. $3x^4 + 11x^2 - 4$
   $(3x^2 - 1)(x^2 + 4)$
34. $2x^4 + 3x^2 - 9$
   $(2x^2 - 3)(x^2 + 3)$
35. $3x^6 + 2x^3 - 8$
   $(3x^3 - 4)(x^3 + 2)$
36. $8x^6 - 10x^3 - 3$
   $(4x^3 + 1)(2x^3 - 3)$

In Exercises 37 to 46, factor each difference of squares over the integers.

37. $x^2 - 9$
   $(x + 3)(x - 3)$
38. $x^2 - 64$
   $(x + 8)(x - 8)$
39. $4a^2 - 49$
   $(2a + 7)(2a - 7)$
40. $81b^2 - 16c^2$
   $(9b + 4c)(9b - 4c)$
41. $1 - 100x^2$
   $(1 + 10x)(1 - 10x)$
42. $1 - 121y^2$
   $(1 + 11y)(1 - 11y)$
43. $x^4 - 9$
   $(x^2 + 3)(x^2 - 3)$
44. $y^4 - 196$
   $(y^2 + 14)(y^2 - 14)$
45. $(x + 5)^2 - 4$
   $(x + 7)(x + 3)$
46. $(x - 3)^2 - 16$
   $(x + 1)(x - 7)$

In Exercises 47 to 54, factor each perfect-square trinomial.

47. $x^2 + 10x + 25$
$(x + 5)^2$

48. $y^2 + 6y + 9$
$(y + 3)^2$

49. $a^2 - 14a + 49$
$(a - 7)^2$

50. $b^2 - 24b + 144$
$(b - 12)^2$

51. $4x^2 + 12x + 9$
$(2x + 3)^2$

52. $25y^2 + 40y + 16$
$(5y + 4)^2$

53. $z^4 + 4z^2w^2 + 4w^4$
$(z^2 + 2w^2)^2$

54. $9x^4 - 30x^2y^2 + 25y^4$
$(3x^2 - 5y^2)^2$

In Exercises 55 to 62, factor each sum or difference of cubes over the integers.

55. $x^3 - 8$
$(x - 2)(x^2 + 2x + 4)$

56. $b^3 + 64$
$(b + 4)(b^2 - 4b + 16)$

57. $8x^3 - 27y^3$
$(2x - 3y)(4x^2 + 6xy + 9y^2)$

58. $64u^3 - 27v^3$
$(4u - 3v)(16u^2 + 12uv + 9v^2)$

59. $8 - x^6$
$(2 - x^2)(4 + 2x^2 + x^4)$

60. $1 + y^{12}$
$(1 + y^4)(1 - y^4 + y^8)$

61. $(x - 2)^3 - 1$
$(x - 3)(x^2 - 3x + 3)$

62. $(y + 3)^3 + 8$
$(y + 5)(y^2 + 4y + 7)$

In Exercises 63 to 68, factor (over the integers) by grouping in pairs.

63. $3x^3 + x^2 + 6x + 2$
$(3x + 1)(x^2 + 2)$

64. $18w^3 + 15w^2 + 12w + 10$
$(6w + 5)(3w^2 + 2)$

65. $ax^2 - ax + bx - b$
$(x - 1)(ax + b)$

66. $a^2y^2 - ay^3 + ac - cy$
$(a - y)(ay^2 + c)$

67. $6w^3 + 4w^2 - 15w - 10$
$(3w + 2)(2w^2 - 5)$

68. $10z^3 - 15z^2 - 4z + 6$
$(2z - 3)(5z^2 - 2)$

In Exercises 69 to 88, use the general factoring strategy to completely factor each polynomial. If the polynomial does not factor, then state that it is nonfactorable over the integers.

69. $18x^2 - 2$
$2(3x - 1)(3x + 1)$

70. $4bx^3 + 32b$
$4b(x + 2)(x^2 - 2x + 4)$

71. $16x^4 - 1$
$(2x - 1)(2x + 1)(4x^2 + 1)$

72. $81y^4 - 16$
$(3y - 2)(3y + 2)(9y^2 + 4)$

73. $12ax^2 - 23axy + 10ay^2$
$a(3x - 2y)(4x - 5y)$

74. $6ax^2 - 19axy - 20ay^2$
$a(6x + 5y)(x - 4y)$

75. $3bx^3 + 4bx^2 - 3bx - 4b$
$b(3x + 4)(x - 1)(x + 1)$

76. $2x^6 - 2$ $2(x - 1) \times (x^2 + x + 1)(x + 1)(x^2 - x + 1)$

77. $72bx^2 + 24bxy + 2by^2$
$2b(6x + y)^2$

78. $64y^3 - 16y^2z + yz^2$
$y(8y - z)^2$

79. $(w - 5)^3 + 8$
$(w - 3)(w^2 - 12w + 39)$

80. $5xy + 20y - 15x - 60$
$5(x + 4)(y - 3)$

81. $x^2 + 6xy + 9y^2 - 1$
$(x + 3y - 1)(x + 3y + 1)$

82. $4y^2 - 4yz + z^2 - 9$
$(2y - z - 3)(2y - z + 3)$

83. $8x^2 + 3x - 4$
Nonfactorable over the integers

84. $16x^2 + 81$
Nonfactorable over the integers

85. $5x(2x - 5)^2 - (2x - 5)^3$
$(2x - 5)^2(3x + 5)$

86. $6x(3x + 1)^3 - (3x + 1)^4$
$(3x + 1)^3(3x - 1)$

87. $4x^2 + 2x - y - y^2$
$(2x - y)(2x + y + 1)$

88. $a^2 + a + b - b^2$
$(a + b)(a - b + 1)$

Connecting Concepts

In Exercises 89 and 90, find all positive values of *k* such that the trinomial is a perfect-square trinomial.

89. $x^2 + kx + 16$
8

90. $36x^2 + kxy + 100y^2$
120

In Exercises 91 and 92, find *k* such that the trinomial is a perfect-square trinomial.

91. $x^2 + 16x + k$
64

92. $x^2 - 14xy + ky^2$
49

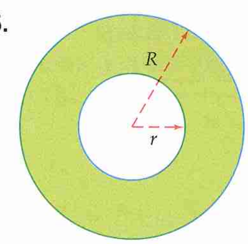
In Exercises 93 and 94, use the general factoring strategy to factor each polynomial. In each exercise, *n* represents a positive integer.

93. $x^{4n} - 1$
$(x^n - 1)(x^n + 1)(x^{2n} + 1)$

94. $x^{4n} - 2x^{2n} + 1$
$(x^n - 1)^2(x^n + 1)^2$

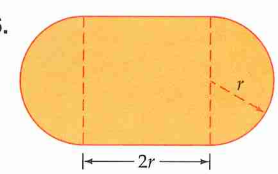
In Exercises 95 to 98, write, in its factored form, the area of the shaded portion of each geometric figure.

95.



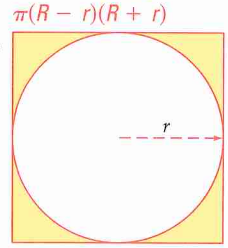
$\pi(R - r)(R + r)$

96.



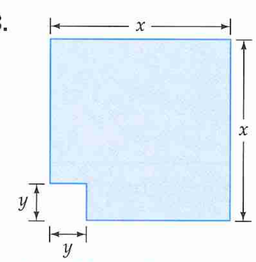
$r^2(\pi + 4)$

97.



$r^2(4 - \pi)$

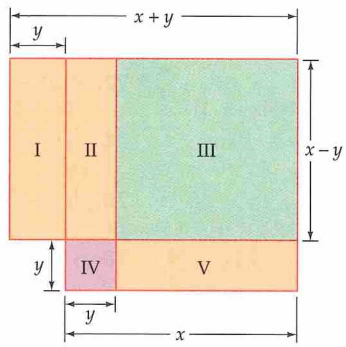
98.



$(x - y)(x + y)$

Projects

1. **GEOMETRY** The ancient Greeks used geometric figures and the concept of area to illustrate many algebraic concepts. The factoring formula $x^2 - y^2 = (x + y)(x - y)$ can be illustrated by the figure below.

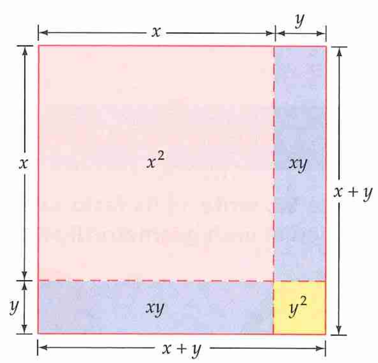


   a. Which regions are represented by $(x + y)(x - y)$?

   b. Which regions are represented by $x^2 - y^2$?

   c. Explain why the area of the regions listed in **a.** must equal the area of the regions listed in **b.**

2. **GEOMETRY** What algebraic formula does the geometric figure below illustrate?



3. **GEOMETRY** Show how the figure below can be used to illustrate the factoring formula for the difference of two cubes.